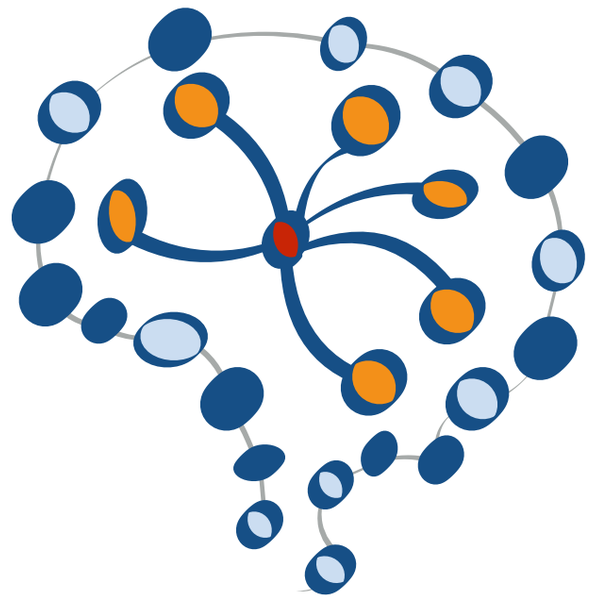
STAT 453: Introduction to Deep Learning and Generative Models



Lecture 05

Fitting Neurons with Gradient Descent

Let's improve upon the perceptron & learn about a neural network model for which the training always converges

Our Goals

- A learning rule that is more robust than the perceptron: ideally converges even if the data are not (linearly) separable.

  ← This lecture

- Combine multiple neurons and layers of neurons ("deep neural nets") to learn more complex decision boundaries (because most real-world problems are not "linear" problems!)

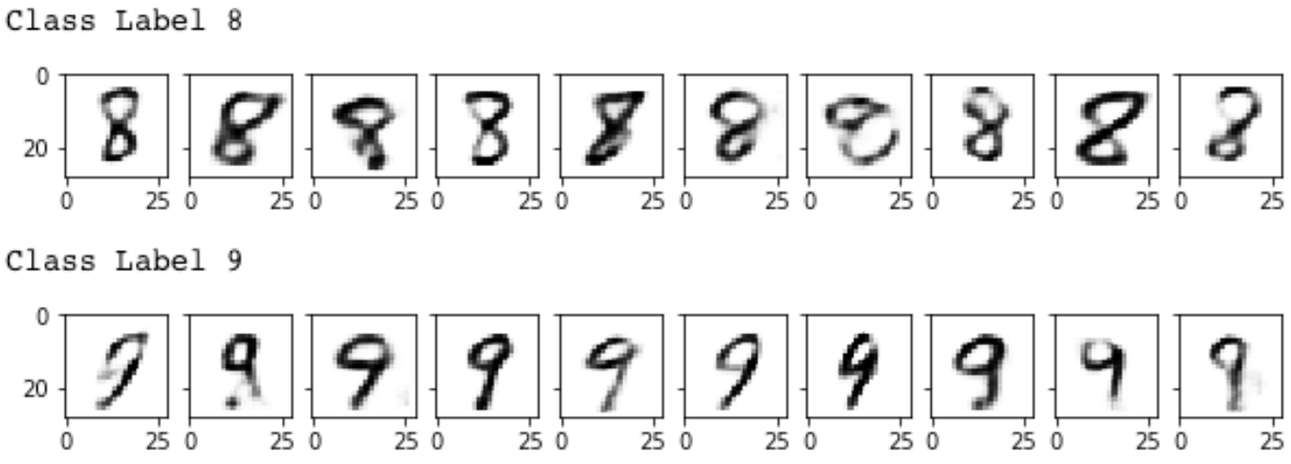
  ← Next lecture(s)

- Handle multiple categories (not just binary) in classification

  ← Next lecture(s)

- Do even fancier things like generating NEW images and text

  ← More towards the end of the course



All based on the same learning algorithm and extensions thereof.
So, this is prob. the most fundamental lecture!

Quiz time

1. The MNIST dataset has 60,000 images of handwritten digits, each of size 28 by 28. What is the dimensions of the 3-order input tensor if we sample a batch of size 64?

2. For classifying each digit into 10 classes (0,1,2…9), what is the dimension of the output tensor for that batch?

Good news:

- To learn how to code, there won't be any "new" mathematical concepts after this lecture.
- Everything in DL will be extensions & applications of these basic concepts.

Scalability is all you need

Lecture Overview

1. Online, batch, and minibatch mode
2. Relation between perceptron and linear regression
3. An iterative training algorithm for linear regression
4. (Optional) Calculus refresher I: Derivatives
5. (Optional) Calculus refresher II: Gradients
6. Understanding gradient descent
7. Training an adaptive linear neuron (Adaline)

Training neural nets: Online, batch, and minibatch modes

1. **Online, batch, and minibatch mode**
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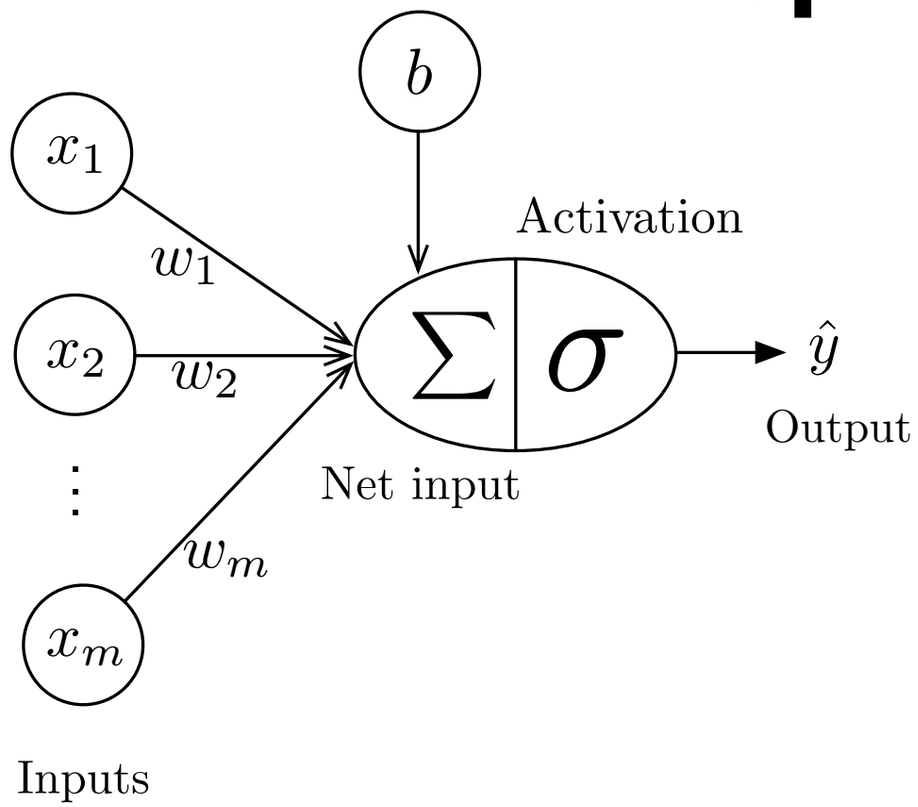
Perceptron Recap



$$\sigma\left(\sum_{i=1}^{m} x_i w_i + b\right) = \sigma\left(\mathbf{x}^T\mathbf{w} + b\right) = \hat{y}$$

$$\sigma(z) = \begin{cases} 0, & z \leq 0 \\ 1, & z > 0 \end{cases}$$

$$b = -\theta$$

Let $\mathcal{D} = (\langle \mathbf{x}^{[1]}, y^{[1]} \rangle, \langle \mathbf{x}^{[2]}, y^{[2]} \rangle, ..., \langle \mathbf{x}^{[n]}, y^{[n]} \rangle) \in (\mathbb{R}^m \times \{0, 1\})^n$

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   - A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$:
     - (a) $\hat{y}^{[i]} := \sigma\left(\mathbf{x}^{[i]T}\mathbf{w} + b\right)$ ← Compute output (prediction)
     - (b) $\text{err} := (y^{[i]} - \hat{y}^{[i]})$ ← Calculate error
     - (c) $\mathbf{w} := \mathbf{w} + err \times \mathbf{x}^{[i]}$, $b := b + err$ ← Update parameters

Quiz time

How is each line of pseudo code translated to a class method in Python?

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m \quad \mathbf{b} := 0$
2. For every training epoch:
   A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$:
      (a) $\hat{y}^{[i]} := \sigma\left(\mathbf{x}^{[i]T}\mathbf{w} + b\right)$
      (b) $\text{err} := (y^{[i]} - \hat{y}^{[i]})$
      (c) $\mathbf{w} := \mathbf{w} + err \times \mathbf{x}^{[i]} \quad b := b + err$

```python
class Perceptron():
    def __init__(self, num_features):
        ...
    def forward(self, x):
        ...
    def backward(self, x, y):
        ...
    def train(self, x, y, epochs):
        ...
    def evaluate(self, x, y):
        ...
```

General Learning Principle

Let $\mathcal{D} = (\langle \mathbf{x}^{[1]}, y^{[1]} \rangle, \langle \mathbf{x}^{[2]}, y^{[2]} \rangle, ..., \langle \mathbf{x}^{[n]}, y^{[n]} \rangle) \in (\mathbb{R}^m \times \{0, 1\})^n$

Targets may be more general than labels

"On-line" mode (aka SGD)

Smarter initialization will be introduced later

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:

A. For **every** $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$:

(a) Compute output (prediction)

(b) Calculate error

(c) Update $\mathbf{w}, b$

This applies to all common neuron models and (deep) neural network architectures!

There are some variants of it, namely the "batch mode" and the "minibatch mode" which we will briefly go over in the next slides and then discuss more later

General Learning Principle

Let $\mathcal{D} = (\langle \mathbf{x}^{[1]}, y^{[1]} \rangle, \langle \mathbf{x}^{[2]}, y^{[2]} \rangle, ..., \langle \mathbf{x}^{[n]}, y^{[n]} \rangle) \in (\mathbb{R}^m \times \{0, 1\})^n$

"On-line" mode

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. **For** every training epoch:
   A. **For** **every** $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$:
      (a) Compute output (prediction)
      (b) Calculate error
      (c) Update $\mathbf{w}, \mathbf{b}$

Batch mode

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. **For** every training epoch:
   A. Take **all** training examples from $\mathcal{D}$:
      (a) Compute output (prediction)
      (b) Calculate error
   B. Update $\mathbf{w}, \mathbf{b}$

General Learning Principle

Let $\mathcal{D} = (\langle \mathbf{x}^{[1]}, y^{[1]} \rangle, \langle \mathbf{x}^{[2]}, y^{[2]} \rangle, ..., \langle \mathbf{x}^{[n]}, y^{[n]} \rangle) \in (\mathbb{R}^m \times \{0, 1\})^n$

"On-line" mode

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In practice, we usually shuffle the dataset prior to each epoch to prevent cycles

Batch mode

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"On-line" mode

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. For **every** $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$:
      (a) Compute output (prediction)
      (b) Calculate error
      (c) Update $\mathbf{w}, \mathbf{b}$

In practice, we usually shuffle the dataset prior to each epoch to prevent cycles

"On-line" mode II (alternative)

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. Pick random $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$:
      (a) Compute output (prediction)
      (b) Calculate error
      (c) Update $\mathbf{w}, \mathbf{b}$

No shuffling required

(Note: strictly speaking, SGD or online learning requires that every training example is used only once, namely number of training epoch = 1)

General Learning Principle

Let $\mathcal{D} = (\langle \mathbf{x}^{[1]}, y^{[1]} \rangle, \langle \mathbf{x}^{[2]}, y^{[2]} \rangle, ..., \langle \mathbf{x}^{[n]}, y^{[n]} \rangle) \in (\mathbb{R}^m \times \{0, 1\})^n$

Minibatch mode

(mix between on-line and batch)

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. **For** every training epoch:

A. **For** every **minibatch** of size k, namely

$\{\langle \mathbf{x}^{[i]}, y^{[i]} \rangle, ..., \langle \mathbf{x}^{[i+k]}, y^{[i+k]} \rangle\} \subset \mathcal{D}$:

(a) Compute output (prediction)

(b) Calculate error

(c) Update $\mathbf{w}, \mathbf{b}$:

$$\mathbf{w} := \mathbf{w} + \Delta\mathbf{w}, \quad \mathbf{b} := \mathbf{b} + \Delta\mathbf{b}$$

The most common mode in deep learning. Any ideas why?

General Learning Principle

Let $\mathcal{D} = (\langle \mathbf{x}^{[1]}, y^{[1]} \rangle, \langle \mathbf{x}^{[2]}, y^{[2]} \rangle, ..., \langle \mathbf{x}^{[n]}, y^{[n]} \rangle) \in (\mathbb{R}^m \times \{0, 1\})^n$

Minibatch mode

(mix between on-line and batch)

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
3. For every minibatch of size *k*:
   - A. Initialize $\Delta \mathbf{w} := 0,\ \Delta b := 0$
   - B. For every $\{\langle \mathbf{x}^{[i]}, y^{[i]} \rangle, ..., \langle \mathbf{x}^{[i+k]}, y^{[i+k]} \rangle\} \subset \mathcal{D}$
     - (a) Compute output (prediction)
     - (b) Calculate error
     - (c) Update $\Delta \mathbf{w}, \Delta b$
   - C. Update $\mathbf{w}, b$:

$$\mathbf{w} := \mathbf{w} + \Delta \mathbf{w}, b := +\Delta b$$

Most commonly used in DL, because

1. Choosing a subset (vs 1 example at a time) takes advantage of vectorization (faster iteration through epoch than on-line)
2. having fewer updates than "on-line" makes updates less noisy
3. makes more updates/ epoch than "batch" and is thus faster

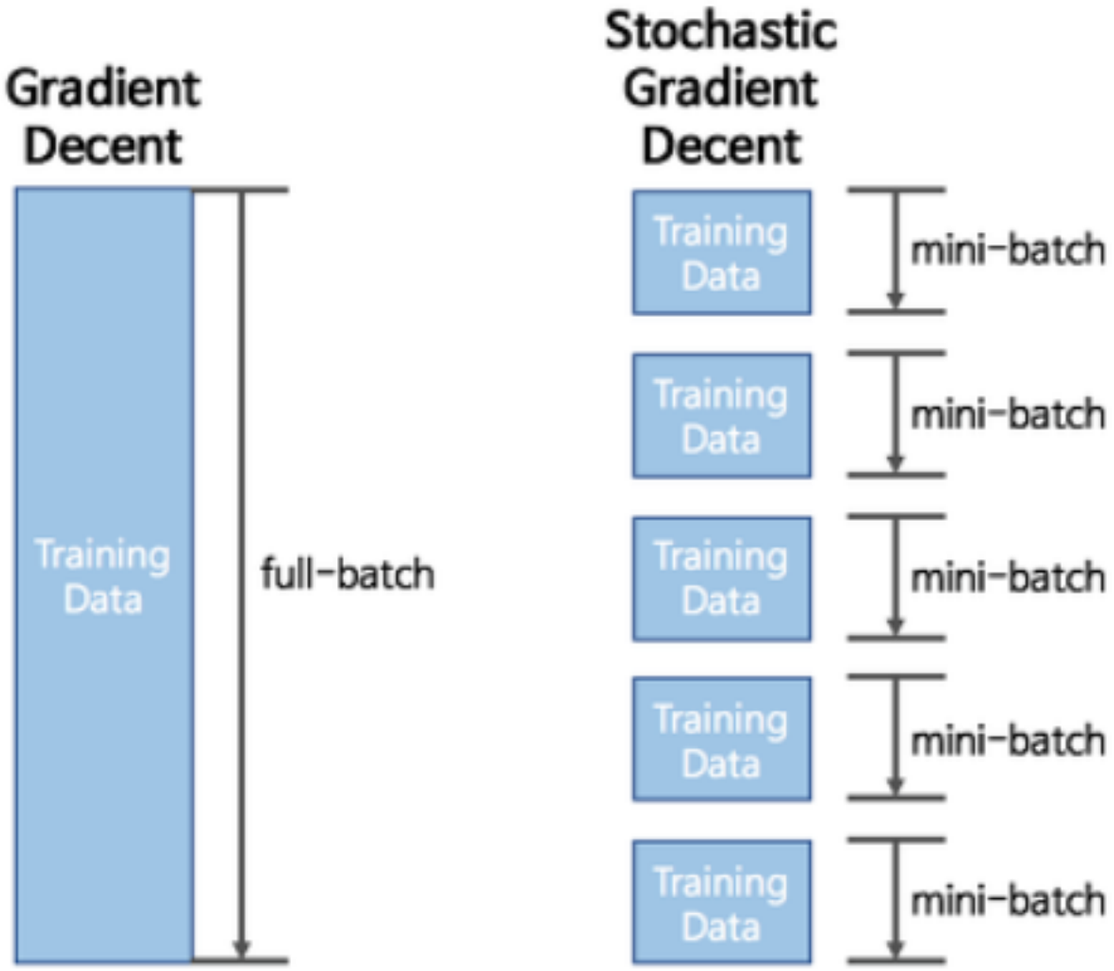
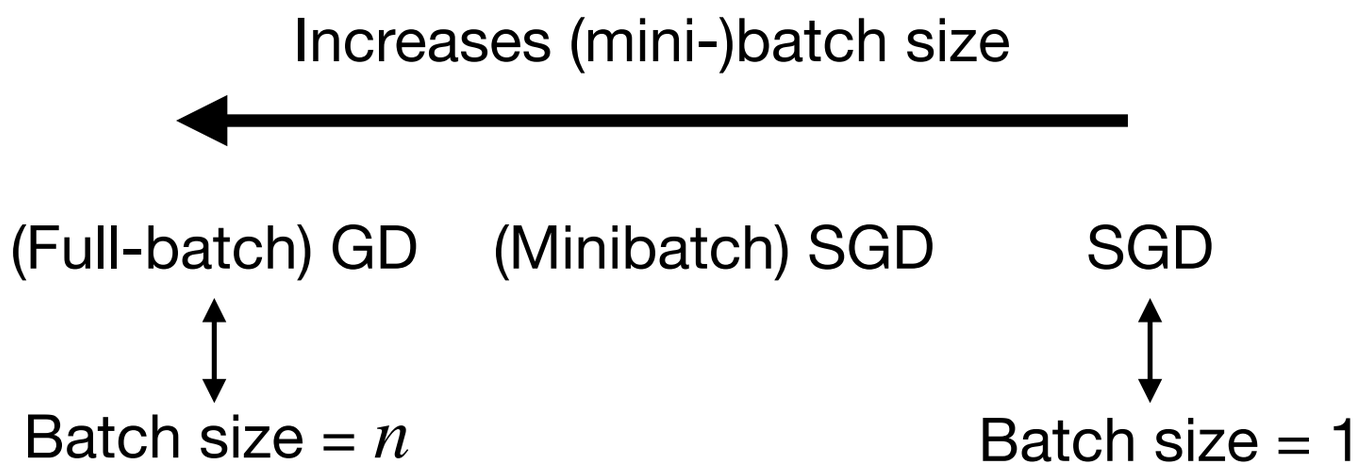


Figure by Jina Yang



Linear regression as a single-layer neural network

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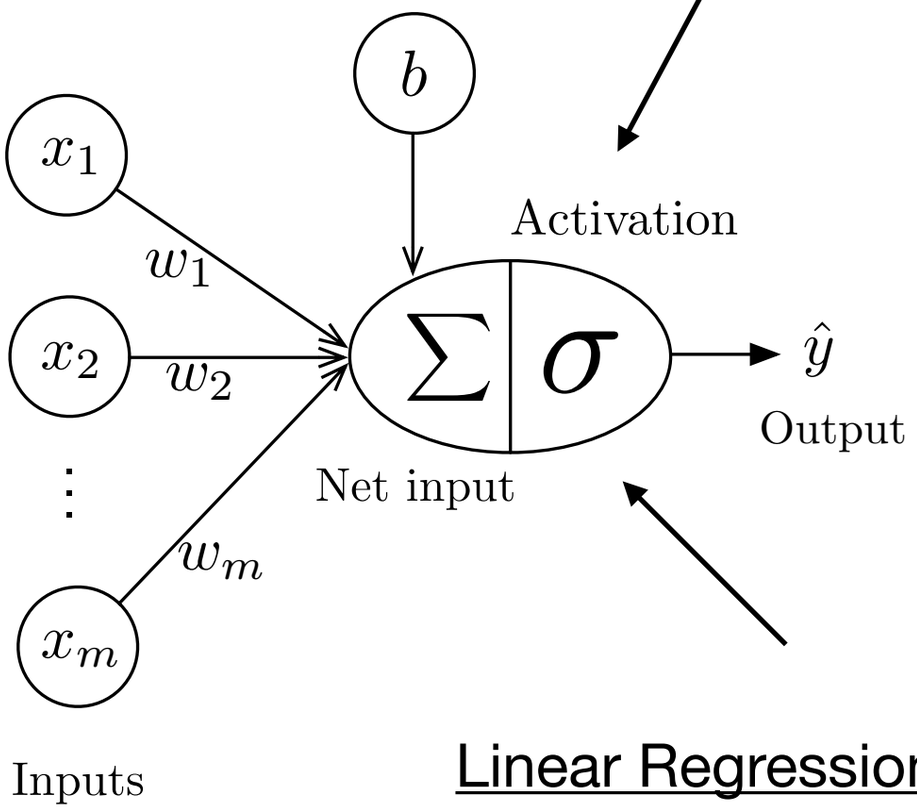
Linear Regression

<u>Perceptron:</u> Activation function is the threshold function

The output is a binary label $\hat{y} \in \{0, 1\}$



<u>Linear Regression:</u> Activation function is the identity function

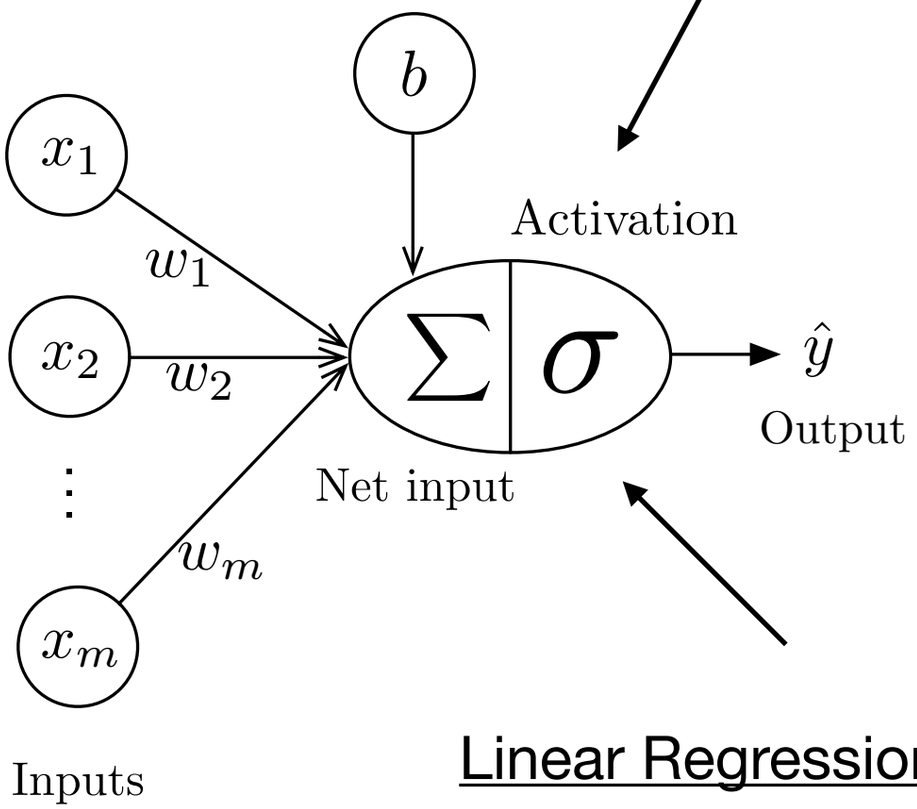
$$\sigma(x) = x$$

The output is a real number $\hat{y} \in \mathbb{R}$

Linear Regression

<u>Perceptron:</u> Activation function is the threshold function

The output is a binary label $\hat{y} \in \{0, 1\}$



You can think of linear regression as a linear neuron!

<u>Linear Regression:</u> Activation function is the identity function

$$\sigma(x) = x$$

The output is a real number $\hat{y} \in \mathbb{R}$

(Least-Squares) Linear Regression

In earlier statistics classes, you probably fit a linear regression model model like this, using the "normal equations" (analytical solution):

$$\mathbf{w} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top y$$

(assuming that the bias is included in $\mathbf{w}$, and the design matrix has an additional vector of 1's)

(Least-Squares) Linear Regression

In earlier statistics classes, you probably fit a model like this:

using the "normal equations:"

$$\mathbf{w} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top y$$

(assuming that the bias is included in $\mathbf{w}$, and the design matrix has an additional vector of 1's)

- Generally, this is the best approach for linear regression (although,
  the matrix inversion might be problematic on large datasets)
- However, we will now learn about another way to learn these parameters iteratively
- Why? Because this is what we will be doing in deep neural nets later, where we have large datasets, many connections, and non-convex loss functions

Training Linear Regression in an iterative fashion

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(Least-Squares) Linear Regression -- iteratively with "brute force"

- A very naive way to fit a linear regression model (and any neural net) is to start with initializing the parameters to 0's or small random values
- Then, for *k* rounds
  - Choose another random set of weights
  - If the model performs better, keep those weights
  - If the model performs worse, discard the weights

This approach is guaranteed to find the optimal solution for very large *k,* but it would be terribly slow.

(Least-Squares) Linear Regression iteratively

- A very naive way to fit a linear regression model (and any neural net) is to start with initializing the parameters to 0's or small random values
- Then, for *k* rounds
  - Choose another random set of weights
  - If the model performs better, keep those weights
  - If the model performs worse, discard the weights

There's a better way!

- We will analyze what effect a change of a parameter has on the predictive performance (loss) of the model
  then, we change the weight a little bit in the direction that improves the performance (minimizes the loss) the most
- We do this in several (small) steps until the loss does not further decrease

(Least-Squares) Linear Regression

The update rule turns out to be this:

"On-line" mode

Perceptron learning rule

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$
   (a) $\hat{y}^{[i]} := \sigma\left(\mathbf{x}^{[i]T}\mathbf{w} + b\right)$
   (b) $\text{err} := (y^{[i]} - \hat{y}^{[i]})$
   (c) $\mathbf{w} := \mathbf{w} + err \times \mathbf{x}^{[i]}$
   $b := b + err$

Stochastic gradient descent

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$
   (a) $\hat{y}^{[i]} := \sigma\left(\mathbf{x}^{[i]T}\mathbf{w} + b\right)$
   (b) $\nabla_{\mathbf{w}}\mathcal{L} = \left(y^{[i]} - \hat{y}^{[i]}\right)\mathbf{x}^{[i]}$
   $\nabla_b\mathcal{L} = \left(y^{[i]} - \hat{y}^{[i]}\right)$
   (c) $\mathbf{w} := \mathbf{w} + \eta \times (-\nabla_{\mathbf{w}}\mathcal{L})$
   $b := b + \eta \times \underbrace{(-\nabla_b\mathcal{L})}$

learning rate

negative gradient

(Least-Squares) Linear Regression

The update rule turns out to be this:

"On-line" mode:

Vectorized

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$
      (a) $\hat{y}^{[i]} := \sigma\left(\mathbf{x}^{[i]T}\mathbf{w} + b\right)$
      (b) $\nabla_{\mathbf{w}}\mathcal{L} = \left(y^{[i]} - \hat{y}^{[i]}\right)\mathbf{x}^{[i]}$
      $\nabla_b\mathcal{L} = \left(y^{[i]} - \hat{y}^{[i]}\right)$
      (c) $\mathbf{w} := \mathbf{w} + \eta \times (-\nabla_{\mathbf{w}}\mathcal{L})$
      $b := b + \eta \times (-\nabla_b\mathcal{L})$

learning rate

negative gradient

$\Longleftrightarrow$

For-Loop (for understanding only)

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$
      (a) $\hat{y}^{[i]} := \sigma\left(\mathbf{x}^{[i]T}\mathbf{w} + b\right)$
   B. For weight *j* in *{1, ..., m}:*
      (b) $\frac{\partial \mathcal{L}}{\partial w_j} = \left(y^{[i]} - \hat{y}^{[i]}\right)x_j^{[i]}$
      (c) $w_j := w_j + \eta \times (-\frac{\partial \mathcal{L}}{\partial w_j})$
   C. $\frac{\partial \mathcal{L}}{\partial b} = \left(y^{[i]} - \hat{y}^{[i]}\right)$
      $b := b + \eta \times (-\frac{\partial \mathcal{L}}{\partial b})$

(Least-Squares) Linear Regression

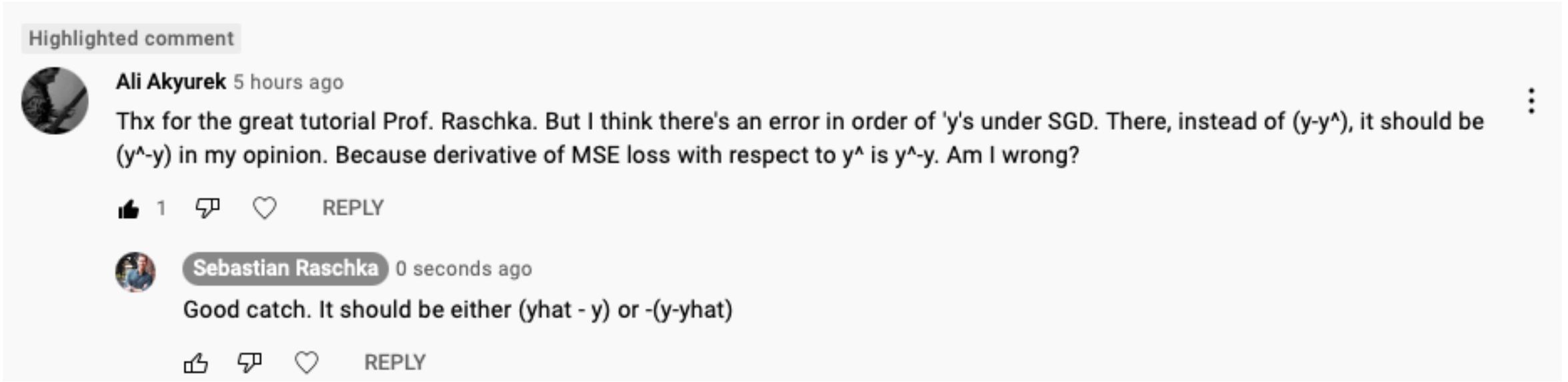
The update rule turns out to be this:

"On-line" mode

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$
      (a) $\hat{y}^{[i]} := \sigma\left(\mathbf{x}^{[i]T}\mathbf{w} + b\right)$
   B. For weight *j* in *{1, ..., m}:*
      (b) $\frac{\partial \mathcal{L}}{\partial w_j} = \left(y^{[i]} - \hat{y}^{[i]}\right) x_j^{[i]}$
      (c) $w_j := w_j + \eta \times (-\frac{\partial \mathcal{L}}{\partial w_j})$
   C. $\frac{\partial \mathcal{L}}{\partial b} = \left(y^{[i]} - \hat{y}^{[i]}\right)$
      $b := b + \eta \times (-\frac{\partial \mathcal{L}}{\partial b})$

Coincidentally, this appears almost to be the same as the perceptron rule, except that the prediction is a **real number** and we have a **learning rate**

Be careful of calculating derivatives…even profs/researchers make mistakes



This learning rule (from the previous slide) is called (stochastic) gradient descent.

So, how did we get there?

First, let's briefly cover relevant background info ...

(Optional section)

Recapping Derivative Rules

1. Online, batch, and minibatch mode
2. Relation between perceptron and linear regression
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4. **(Optional) Calculus refresher I: Derivatives**
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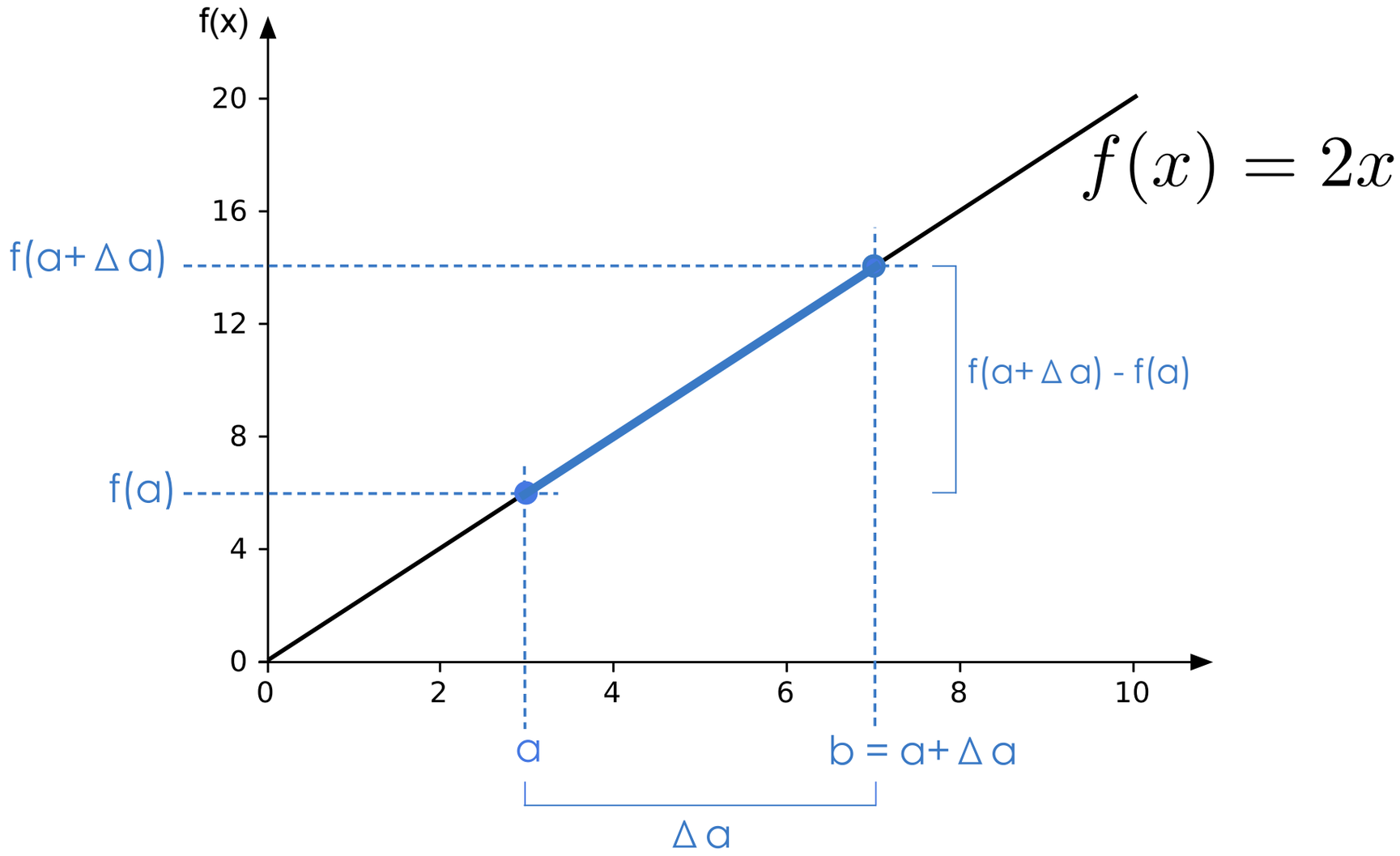
Differential Calculus Refresher

Derivative of a function = "rate of change" = "slope"



$$\text{Slope} = \frac{f(a+\Delta a) - f(a)}{a + \Delta a - a} = \frac{f(a+\Delta a) - f(a)}{\Delta a}$$

Function Derivative

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example 1: $f(x) = 2x$

$$\begin{aligned}
\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x + 2\Delta x - 2x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2.
\end{aligned}$$

Numerical vs Analytical/Symbolical Derivatives

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

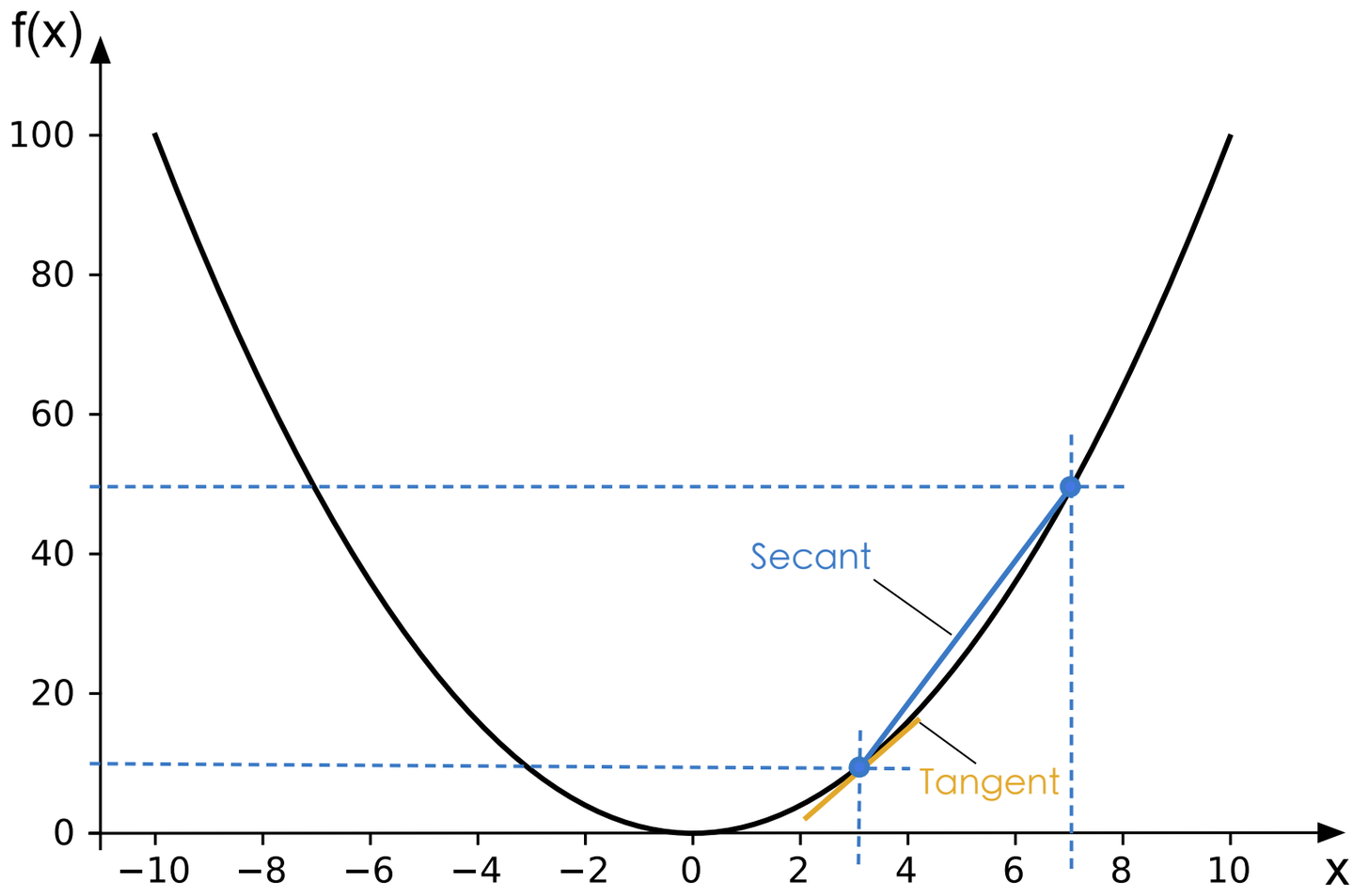
Example 2: $f(x) = x^2$

$$\begin{aligned}
\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \Delta x.
\end{aligned}$$

Numerical vs Analytical/Symbolical Derivatives

Conceptually, we obtained the derivative $\frac{d}{dx}x^2 = 2x$

By approximating the slope (tangent) by a secant between two points (as before)



A Cheatsheet for You (1)

	Function $f(x)$	Derivative with respect to x
1	a	0
2	x	1
3	ax	a
4	x^2	$2x$
5	x^a	ax^{a-1}
6	a^x	$\log(a)a^x$
7	$\log(x)$	$1/x$
8	$\log_a(x)$	$1/(x\log(a))$
9	$\sin(x)$	$\cos(x)$
10	$\cos(x)$	$-\sin(x)$
11	$\tan(x)$	$\sec^2(x)$

A Cheatsheet for You (2)

	Function	Derivative
Sum Rule	$f(x) + g(x)$	$f'(x) + g'(x)$
Difference Rule	$f(x) - g(x)$	$f'(x) - g'(x)$
Product Rule	$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$f(x)/g(x)$	$[g(x)f'(x) - f(x)g'(x)]/[g(x)]^2$
Reciprocal Rule	$1/f(x)$	$-[f'(x)]/[f(x)]^2$
Chain Rule	$f(g(x))$	$f'(g(x))g'(x)$

Chain Rule

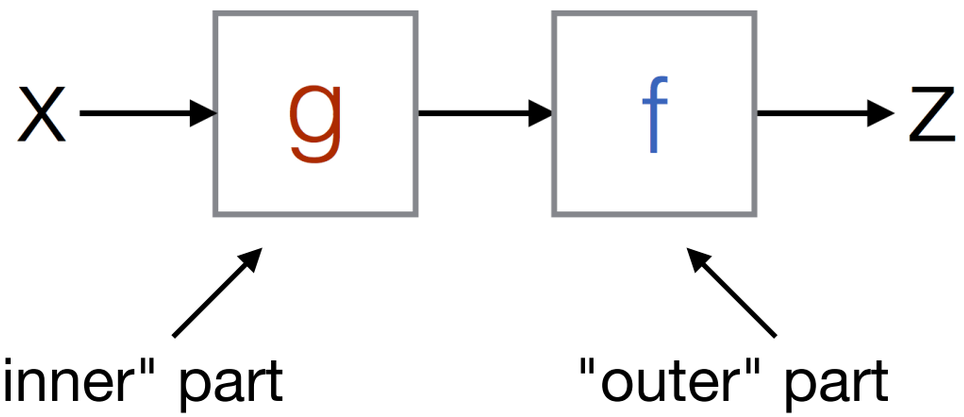
- The chain rule is basically the essence of training (deep) neural networks
- If you understand and learn how to apply the chain rule to various function decompositions, deep learning will be super easy and even seem trivial to you from now on
- In fact, neural networks will become even easier to understand than any algorithm you learned about in my previous ML class

Scalability is all you need: general purpose, scalable

Chain Rule & "Computation Graph" Intuition

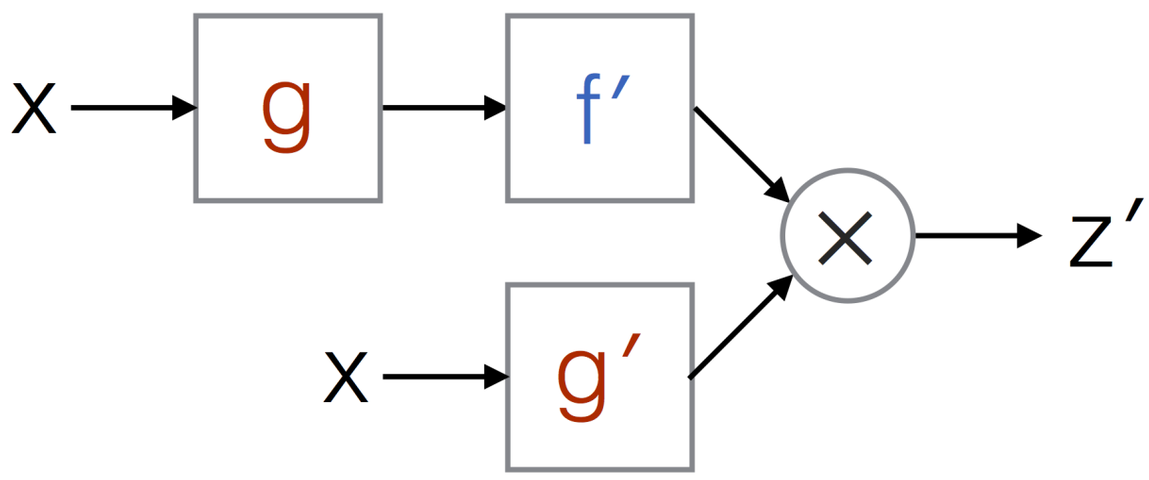
$$F(x) = f(\,g(x)\,) = z$$

Decomposition of some (nested) function:



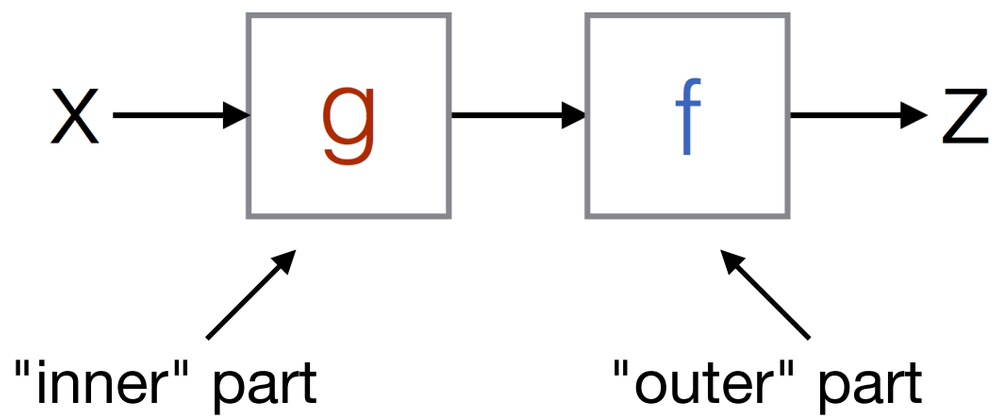
$$F'(x) = f'(\,g(x)\,)\,g'(x) = z'$$

Derivative of that nested function:



Chain Rule & "Computation Graph" Intuition

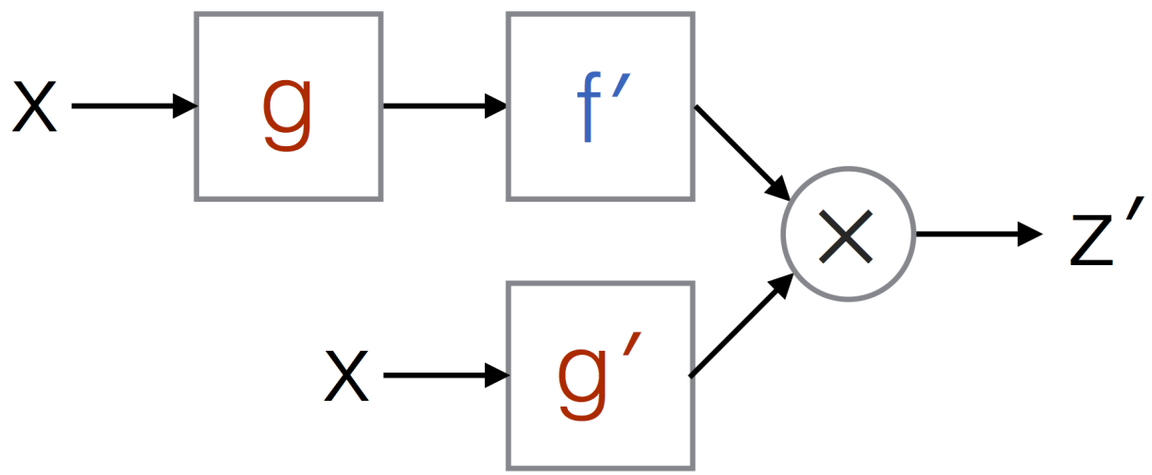
$$F(x) = f(\,g(x)\,) = z$$



Later, we will see that PyTorch can do that automatically for us :) (PyTorch literally keeps a computation graph in the background)

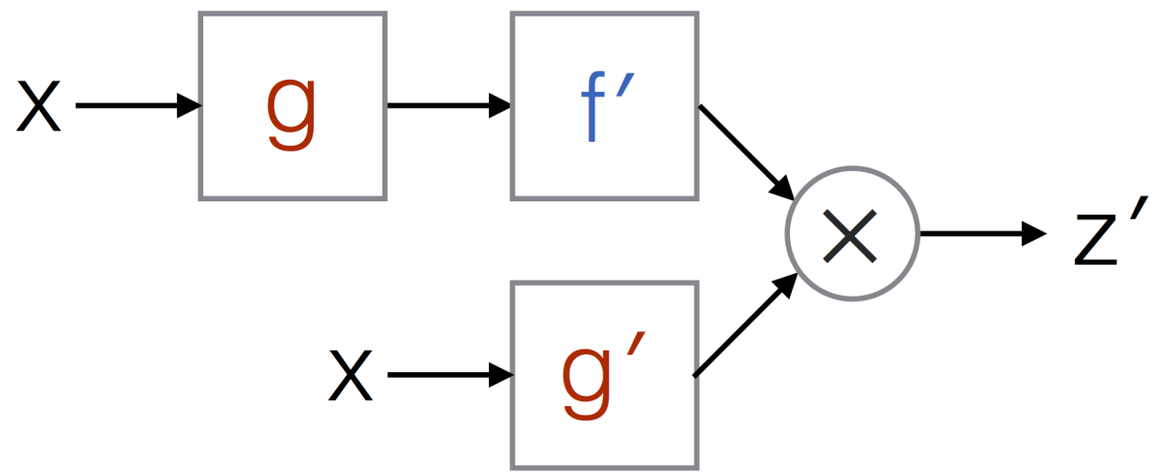
Also, PyTorch can compute the derivatives of most (differentiable) functions automatically

$$F'(x) = f'(\,g(x)\,)\,g'(x) = z'$$



Chain Rule & "Computation Graph" Intuition

$$F'(x) = f'(\ g(x)\ )\ g'(x) = z'$$



In text, for efficiency, we will mostly use the Leibniz notation:

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Chain Rule Example

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Example: $f(x) = \log(\sqrt{x})$

substituting $\frac{df}{dx} = \frac{d}{dg}\log(g) \cdot \frac{d}{dx}\sqrt{x}$

with $\frac{d}{dg}\log(g) = \frac{1}{g} = \frac{1}{\sqrt{x}}$ and $\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

leads us to the solution $\frac{df}{dx} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$

Chain Rule for Arbitrarily Long Function Compositions

$$F(x) = f(g(h(u(v(x)))))$$

$$\frac{dF}{dx} = \frac{d}{dx}F(x) = \frac{d}{dx}f(g(h(u(v(x)))))$$

$$= \frac{df}{dg} \cdot \frac{dg}{dh} \cdot \frac{dh}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Chain Rule for Arbitrarily Long Function Compositions

$$\frac{dF}{dx} = \frac{d}{dx}F(x) = \frac{d}{dx}f(g(h(u(v(x)))))$$
$$= \frac{df}{dg} \cdot \frac{dg}{dh} \cdot \frac{dh}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Also called "reverse mode" as we start with the outer function. In neural nets, this will be from right to left.

We could also start from the inner parts ("forward mode")

$$\frac{dv}{dx} \cdot \frac{du}{dv} \cdot \frac{dh}{du} \cdot \frac{dg}{dh} \cdot \frac{df}{dg}$$

- Backpropagation (covered later) is basically "reverse" mode auto-differentiation
- It is cheaper than forward mode if we work with gradients, since then we have matrix-"vector" multiplications instead of matrix multiplications

Gradients: Derivatives of Multivariable Functions

1. Online, batch, and minibatch mode
2. Relation between perceptron and linear regression
3. An iterative training algorithm for linear regression
4. (Optional) Calculus refresher I: Derivatives
5. **(Optional) Calculus refresher II: Gradients**
6. Understanding gradient descent
7. Training an adaptive linear neuron (Adaline)

Gradients: Derivatives of Multivariable* Functions

***note that in some fields, the terms "multivariable" and "multivariate" are used interchangeably,**
but here, we really mean "multivariable" because "multivariate" means "multiple outputs", which is
not the case here -- similarly, in most DL applications output one prediction value, or one prediction value per training example

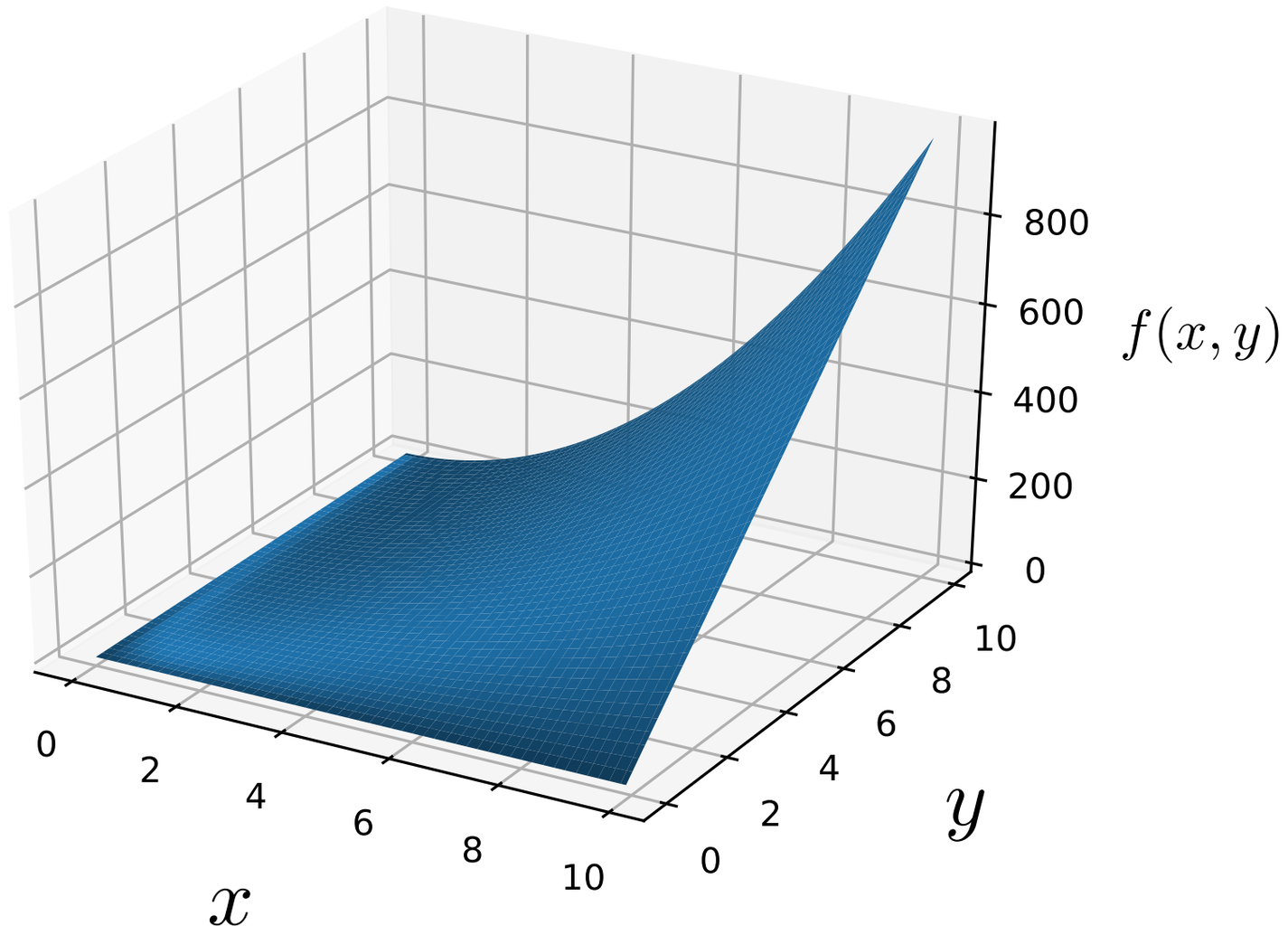
$$f(x, y, z, ...)$$

$$\nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \\ \vdots \end{bmatrix}$$

For gradients, we use the "partial" symbol to denote partial derivatives; more of a notational convention and the concept is the same as before when we were computing ordinary derivatives (denoted them as "d")

Gradients: Derivatives of Multivariable Functions

Example: $f(x, y) = x^2 y + y$



Gradients: Derivatives of Multivariable Functions

Example: $f(x, y) = x^2 y + y$

$$\nabla f(x, y) = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{bmatrix},$$

where

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} x^2 y + y = 2xy$$

(via the power rule and constant rule), and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} x^2 y + y = x^2 + 1.$$

So, the gradient of the function f is defined as

$$\nabla f(x, y) = \begin{bmatrix} 2xy \\ x^2 + 1 \end{bmatrix}.$$

Gradients & the Multivariable Chain Rule

Suppose we have a composite function like this:

$$f\big(g(x), h(x)\big)$$

Remember the regular chain rule for a single input:

$$\frac{d}{dx}\big[f(g(x))\big] = \frac{df}{dg} \cdot \frac{dg}{dx}$$

For two inputs, we now have

$$\frac{d}{dx}\big[f(g(x), h(x))\big] = \frac{\partial f}{\partial g} \cdot \frac{dg}{dx} + \frac{\partial f}{\partial h} \cdot \frac{dh}{dx}$$

Gradients & the Multivariable Chain Rule

$$f\big(g(x), h(x)\big)$$

$$\frac{d}{dx}\big[f(g(x), h(x))\big] = \frac{\partial f}{\partial g} \cdot \frac{dg}{dx} + \frac{\partial f}{\partial h} \cdot \frac{dh}{dx}$$

Example:

$$f\big(g, h\big) = g^2 h + h$$

where $g(x) = 3x,$ and $h(x) = x^2$

$$\frac{\partial f}{\partial g} = 2gh \qquad \frac{\partial f}{\partial h} = g^2 + 1$$

$$\frac{dg}{dx} = \frac{d}{dx} 3x = 3 \qquad \frac{dh}{dx} = \frac{d}{dx} x^2 = 2x$$

$$\begin{aligned}\frac{d}{dx}\big[f(g(x))\big] &= [2gh \cdot 3] + [(g^2 + 1) \cdot 2x] \\ &= 2xg^2 + 6gh + 2x\end{aligned}$$

Gradients & the Multivariable Chain Rule in Vector Form

$$f\big(g(x), h(x)\big)$$

$$\begin{aligned}\frac{d}{dx}\big[f(g(x), h(x))\big] &= \frac{\partial f}{\partial g}\cdot\frac{dg}{dx} + \frac{\partial f}{\partial h}\cdot\frac{dh}{dx} \\ &= \nabla f \cdot \mathbf{v}'(x).\end{aligned}$$

Where

$$\mathbf{v}(x) = \begin{bmatrix} g(x) \\ h(x) \end{bmatrix} \qquad \mathbf{v}'(x) = \frac{d}{dx}\begin{bmatrix} g(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} dg/dx \\ dh/dx \end{bmatrix}$$

Putting it together:

$$\nabla f \cdot \mathbf{v}'(x) = \begin{bmatrix} \partial f/\partial g \\ \partial f/\partial h \end{bmatrix} \cdot \begin{bmatrix} dg/dx \\ dh/dx \end{bmatrix} = \frac{\partial f}{\partial g}\cdot\frac{dg}{dx} + \frac{\partial f}{\partial h}\cdot\frac{dh}{dx}$$

The Jacobian (Matrix)

$$\mathbf{f}(x_1, x_2, ..., x_m) = \begin{bmatrix} f_1\left(x_1, x_2, x_3, \cdots x_m\right) \\ f_2\left(x_1, x_2, x_3, \cdots x_m\right) \\ f_3\left(x_1, x_2, x_3, \cdots x_m\right) \\ \vdots \\ f_m\left(x_1, x_2, x_3, \cdots x_m\right) \end{bmatrix}$$

$$J\left(x_1, x_2, x_3, \cdots x_m\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \cdots & \frac{\partial f_3}{\partial x_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \frac{\partial f_m}{\partial x_3} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix}$$

The Jacobian (Matrix)

$$\mathbf{f}(x_1, x_2, ..., x_m) = \begin{bmatrix} f_1\left(x_1, x_2, x_3, \cdots x_m\right) \\ f_2\left(x_1, x_2, x_3, \cdots x_m\right) \\ f_3\left(x_1, x_2, x_3, \cdots x_m\right) \\ \vdots \\ f_m\left(x_1, x_2, x_3, \cdots x_m\right) \end{bmatrix}$$

$$J\left(x_1, x_2, x_3, \cdots x_m\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \cdots & \frac{\partial f_3}{\partial x_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \frac{\partial f_m}{\partial x_3} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix} \left(\nabla f_1\right)^\top$$

Second Order Derivatives

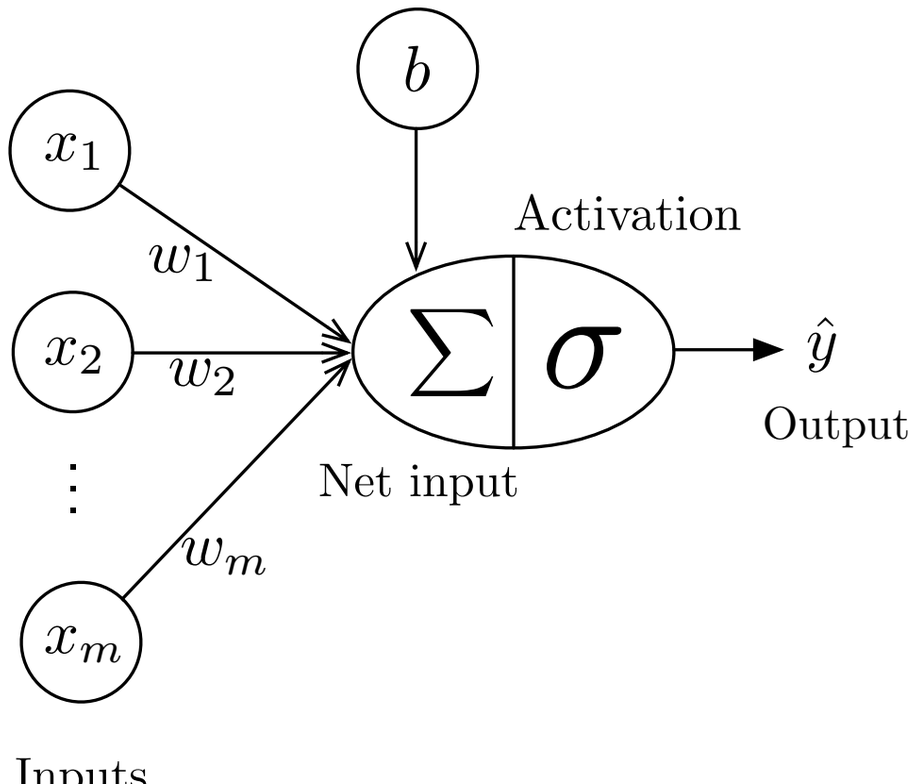
Scalability is all you need

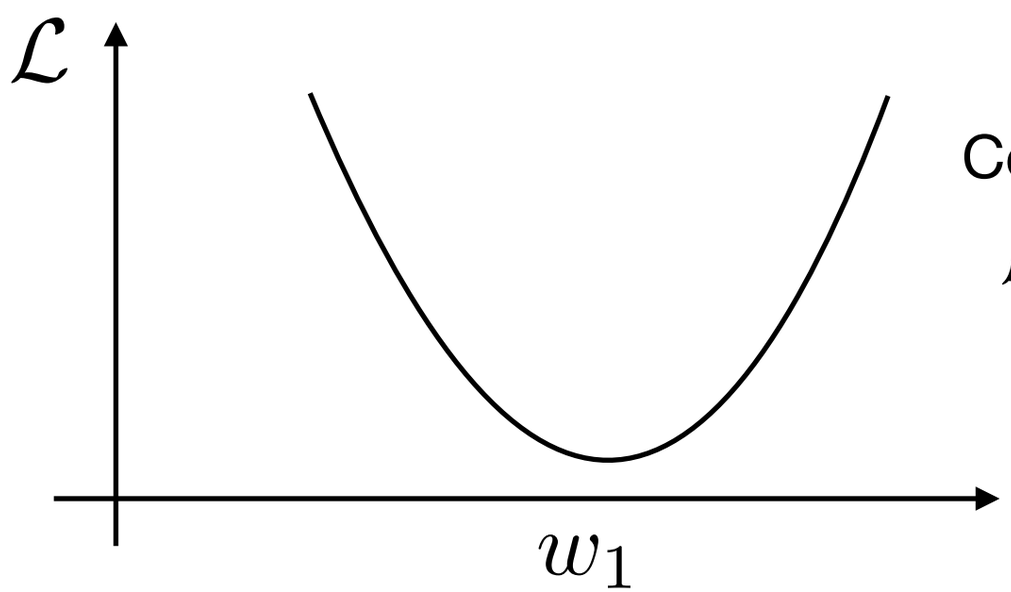
Lucky for you, we won't need second order derivatives in this class ;)

Training Linear Regression with Gradient Descent

1. Online, batch, and minibatch mode
2. Relation between perceptron and linear regression
3. An iterative training algorithm for linear regression
4. (Optional) Calculus refresher I: Derivatives
5. (Optional) Calculus refresher II: Gradients
6. **Understanding gradient descent**
7. Training an adaptive linear neuron (Adaline)

Back to Linear Regression



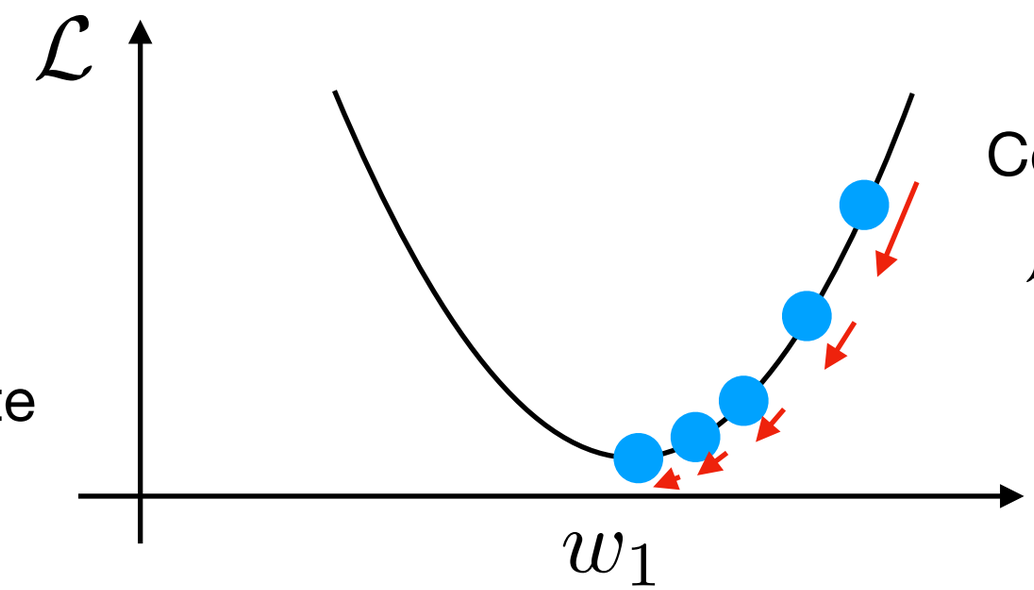


Convex loss function

$$\mathcal{L}(\mathbf{w}, b) = \sum_i (\hat{y}^{[i]} - y^{[i]})^2$$

Gradient Descent

Learning rate and steepness of the gradient determine how much we update



Convex loss function

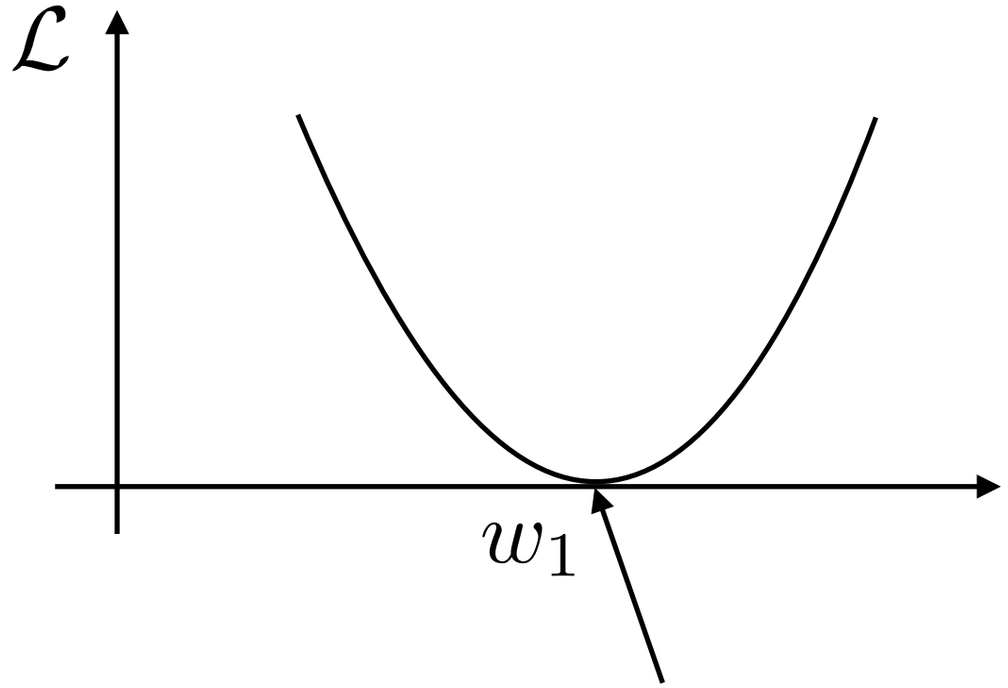
$$\mathcal{L}(\mathbf{w}, b) = \sum_i (\hat{y}^{[i]} - y^{[i]})^2$$

***Q**: what is a convex function? Definition?*

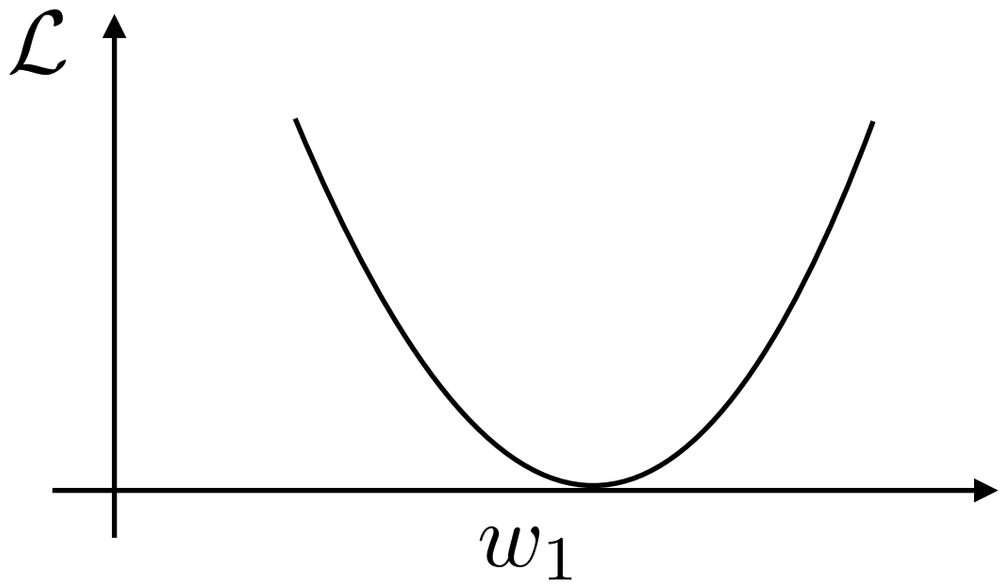
MSE loss is convex loss

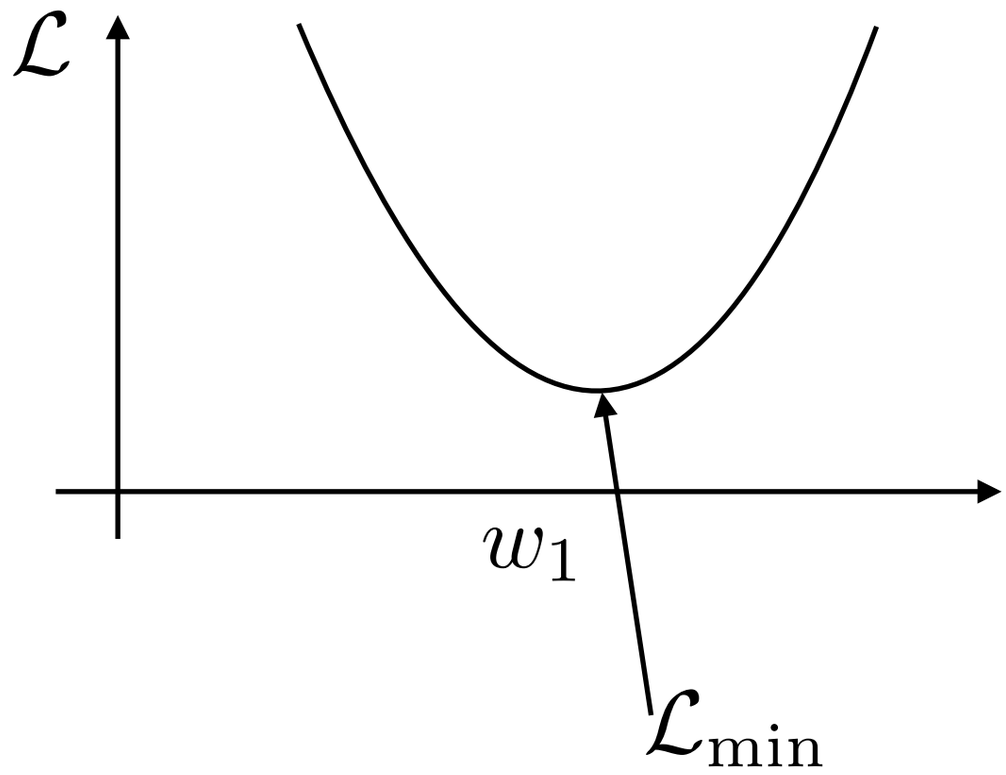
Why is the Loss convex?
Because we assumed it's the
Sum Squared Error (SSE) or Mean Squared Error (MSE)
(Not every loss is convex.)

$$SSE(y, \hat{y}) = \sum_i (y - \hat{y})^2$$



Benefits of convexity





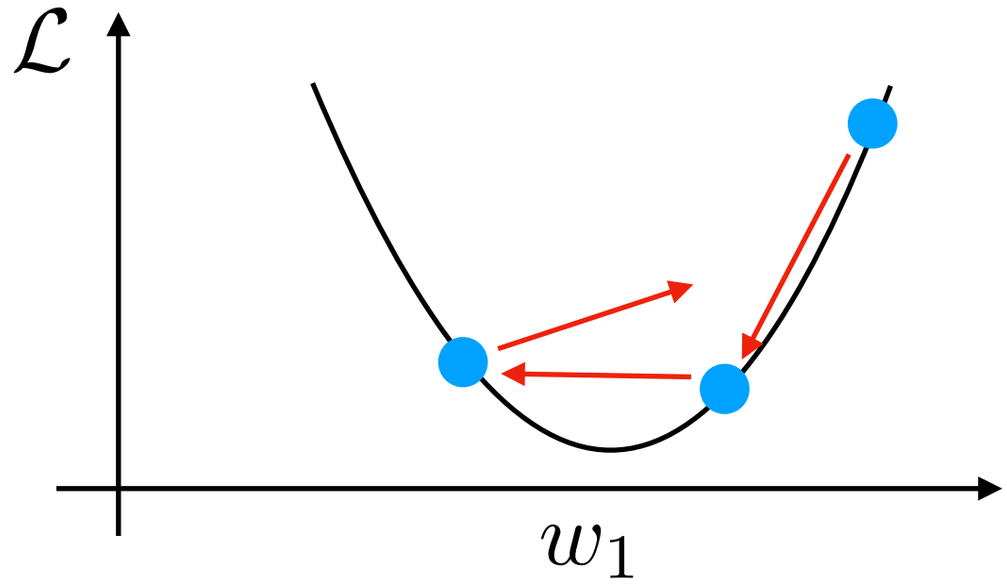
Esp. for linear models, it is often not possible to achieve a zero loss even on the training data

Theoretical guarantee of convexity: reaching a unique minimizer efficiently

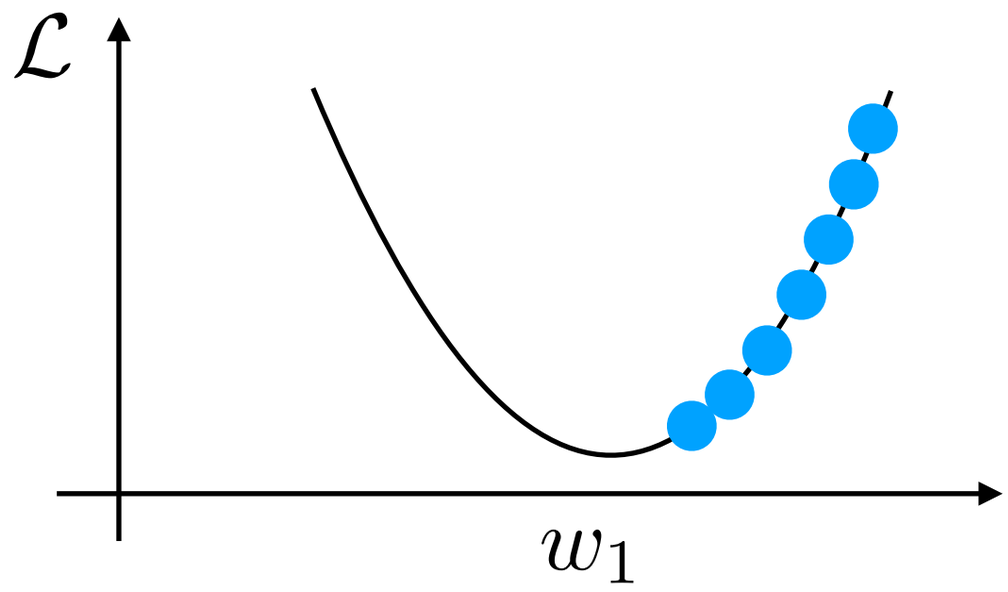
For general neural networks (due to nonlinear activation), no guarantee

Gradient Descent

If the learning rate is too large, we can overshoot



If the learning rate is too small, convergence is very slow



(Least-Squares) Linear Regression

The update rule turns out to be this:

"On-line" mode

Perceptron learning rule

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$
   (a) $\hat{y}^{[i]} := \sigma\big(\mathbf{x}^{[i]T}\mathbf{w} + b\big)$
   (b) $\text{err} := (y^{[i]} - \hat{y}^{[i]})$
   (c) $\mathbf{w} := \mathbf{w} + err \times \mathbf{x}^{[i]}$
   $b := b + err$

Stochastic gradient descent

1. Initialize $\mathbf{w} := \mathbf{0} \in \mathbb{R}^m$, $\mathbf{b} := 0$
2. For every training epoch:
   A. For every $\langle \mathbf{x}^{[i]}, y^{[i]} \rangle \in \mathcal{D}$
   (a) $\hat{y}^{[i]} := \sigma\big(\mathbf{x}^{[i]T}\mathbf{w} + b\big)$
   (b) $\nabla_{\mathbf{w}}\mathcal{L} = -\big(y^{[i]} - \hat{y}^{[i]}\big)\mathbf{x}^{[i]}$
   $\nabla_b\mathcal{L} = -\big(y^{[i]} - \hat{y}^{[i]}\big)$
   (c) $\mathbf{w} := \mathbf{w} + \eta \times (-\nabla_{\mathbf{w}}\mathcal{L})$
   $b := b + \eta \times \underbrace{(-\nabla_b\mathcal{L})}$

learning rate

negative gradient

Linear Regression Loss Derivative

$$\mathcal{L}(\mathbf{w}, b) = \sum_i (\hat{y}^{[i]} - y^{[i]})^2$$

Sum of Squared Error (SSE) loss

Also called Squared Loss

$$\frac{\partial \mathcal{L}}{\partial w_j} = \frac{\partial}{\partial w_j} \sum_i (\hat{y}^{[i]} - y^{[i]})^2$$

$$= \frac{\partial}{\partial w_j} \sum_i (\sigma(\mathbf{w}^T \mathbf{x}^{[i]}) - y^{[i]})^2$$

$$= \sum_i 2(\sigma(\mathbf{w}^T \mathbf{x}^{[i]}) - y^{[i]}) \frac{\partial}{\partial w_j} (\sigma(\mathbf{w}^T \mathbf{x}^{[i]}) - y^{[i]})$$

$$= \sum_i 2(\sigma(\mathbf{w}^T \mathbf{x}^{[i]}) - y^{[i]}) \frac{d\sigma}{d(\mathbf{w}^T \mathbf{x}^{[i]})} \frac{\partial}{\partial w_j} \mathbf{w}^T \mathbf{x}^{[i]}$$

$$= \sum_i 2(\sigma(\mathbf{w}^T \mathbf{x}^{[i]}) - y^{[i]}) \frac{d\sigma}{d(\mathbf{w}^T \mathbf{x}^{[i]})} x_j^{[i]}$$

(Note that the activation function is the identity function in linear regression)

$$= \sum_i 2(\sigma(\mathbf{w}^T \mathbf{x}^{[i]}) - y^{[i]}) x_j^{[i]}$$

Linear Regression Loss Derivative (alt.)

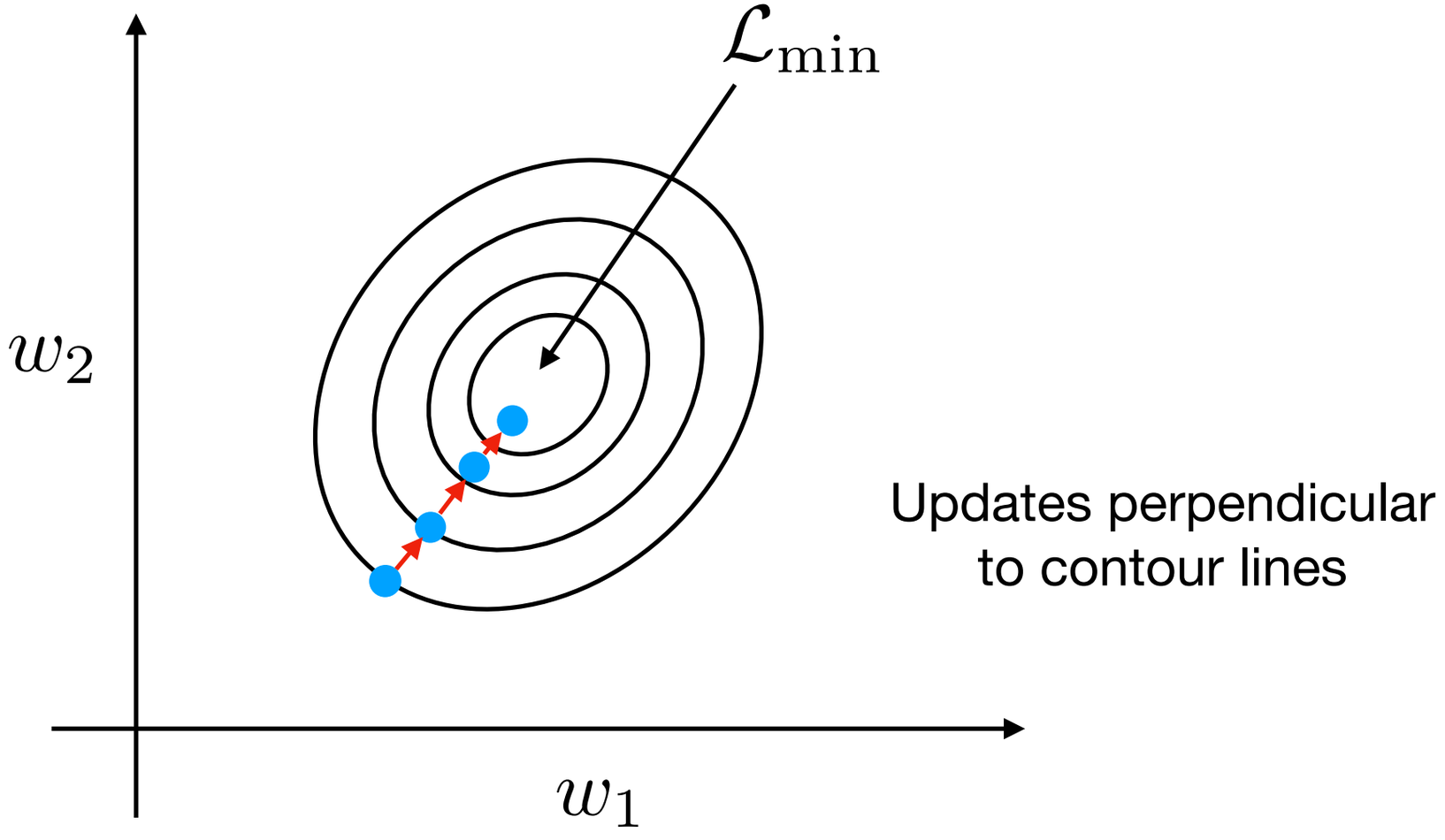
$$\mathcal{L}(\mathbf{w}, b) = \frac{1}{2n}\sum_i (\hat{y}^{[i]} - y^{[i]})^2$$

Mean Squared Error (MSE) loss often scaled by factor 1/2 for convenience

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial w_j} &= \frac{\partial}{\partial w_j}\frac{1}{2n}\sum_i (\hat{y}^{[i]} - y^{[i]})^2 \\
&= \frac{\partial}{\partial w_j}\sum_i \frac{1}{2n}(\sigma(\mathbf{w}^T\mathbf{x}^{[i]}) - y^{[i]})^2 \\
&= \sum_i \frac{1}{n}(\sigma(\mathbf{w}^T\mathbf{x}^{[i]}) - y^{[i]})\frac{\partial}{\partial w_j}(\sigma(\mathbf{w}^T\mathbf{x}^{[i]}) - y^{[i]}) \\
&= \frac{1}{n}\sum_i (\sigma(\mathbf{w}^T\mathbf{x}^{[i]}) - y^{[i]})\frac{d\sigma}{d(\mathbf{w}^T\mathbf{x}^{[i]})}\frac{\partial}{\partial w_j}\mathbf{w}^T\mathbf{x}^{[i]} \\
&= \frac{1}{n}\sum_i (\sigma(\mathbf{w}^T\mathbf{x}^{[i]}) - y^{[i]})\frac{d\sigma}{d(\mathbf{w}^T\mathbf{x}^{[i]})}x_j^{[i]} \\
&= \frac{1}{n}\sum_i (\sigma(\mathbf{w}^T\mathbf{x}^{[i]}) - y^{[i]})x_j^{[i]}
\end{aligned}$$

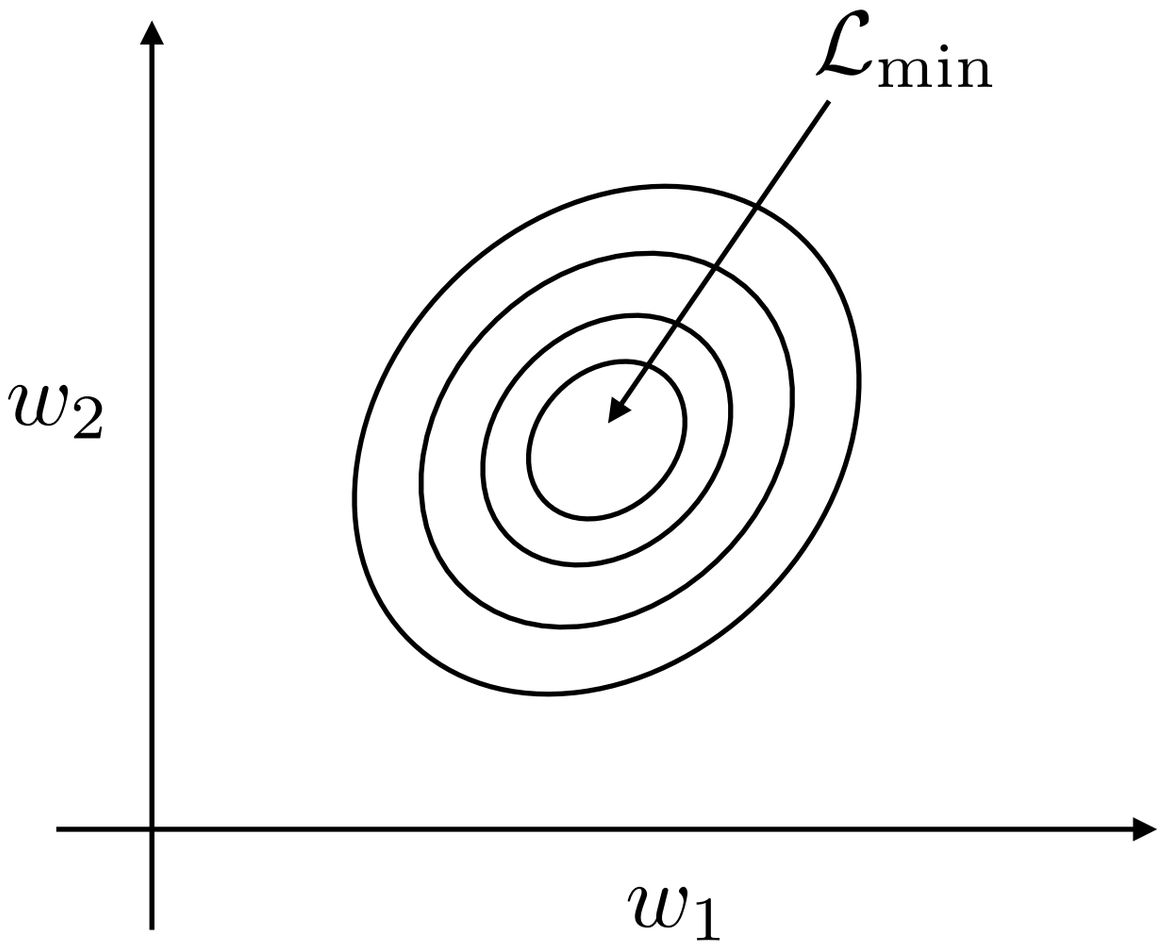
(Note that the activation function is the identity function in linear regression)

Batch Gradient Descent as Surface Plot



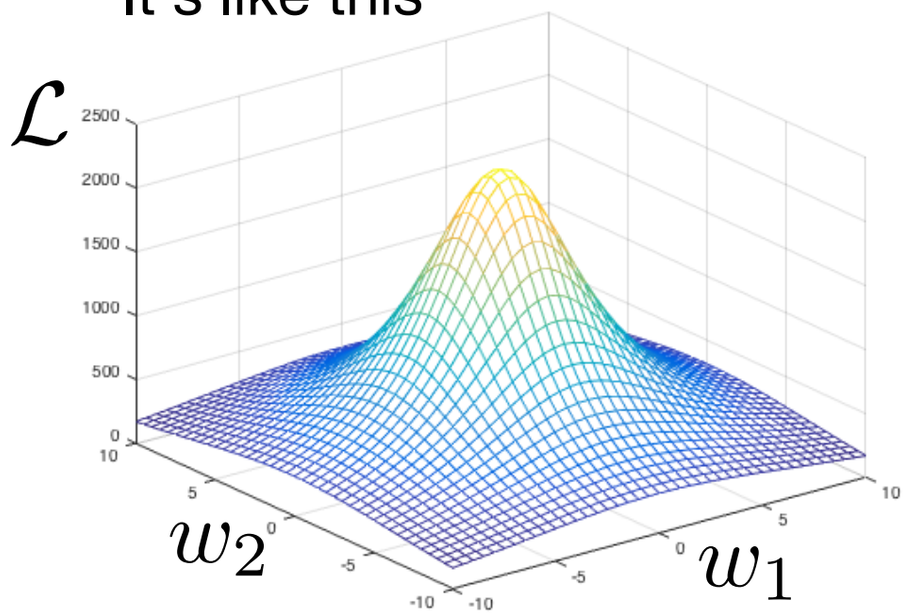
How to think about contour plots?

What does this figure mean/show?

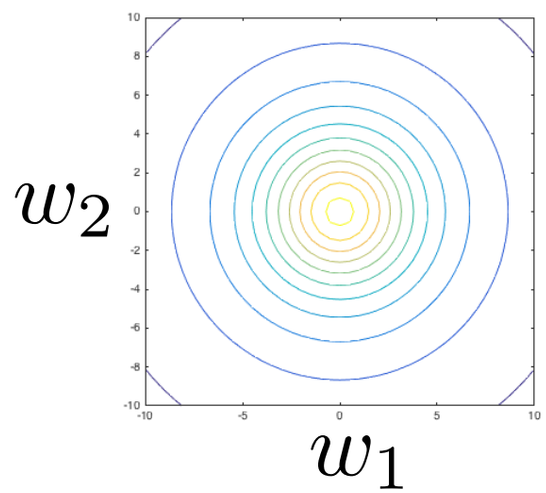


It's simply showing the loss plot (previous slide) for 2 instead of 1 weight

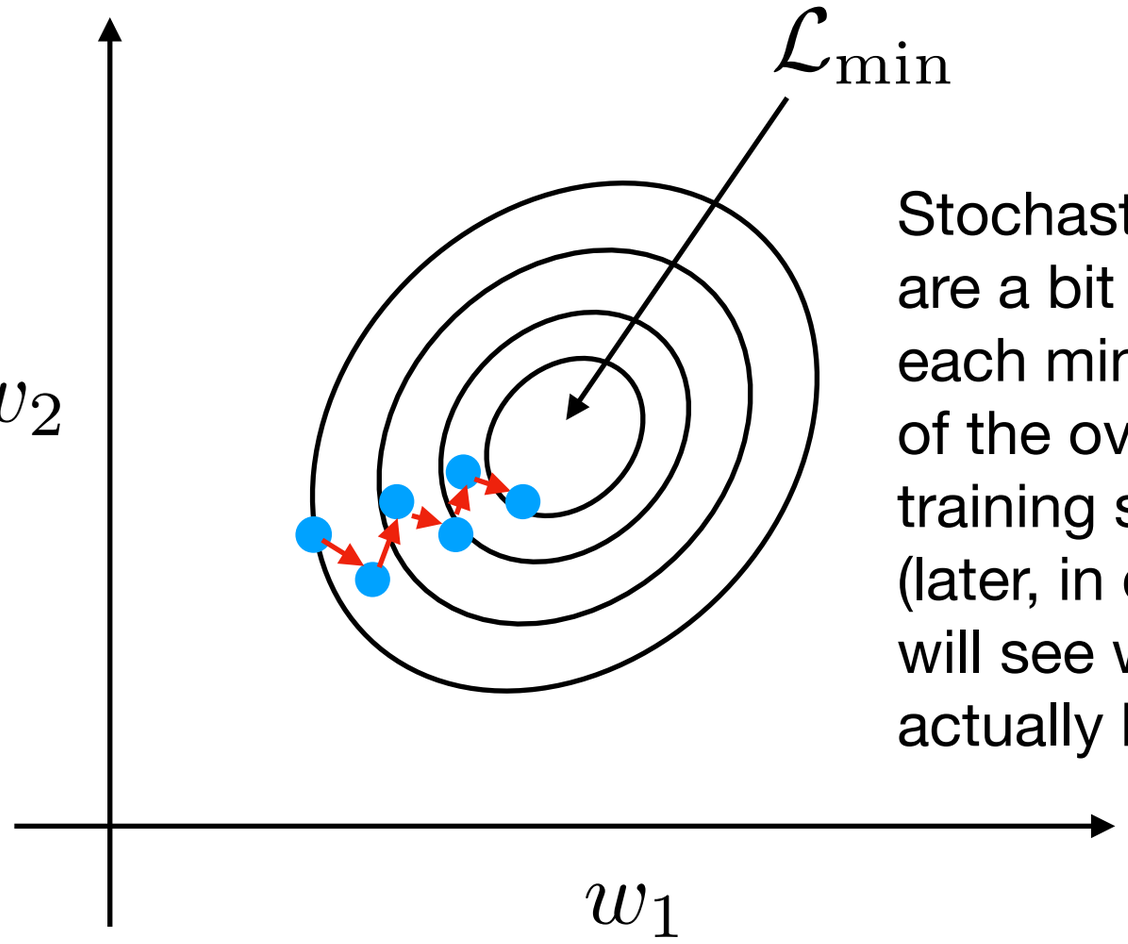
It's like this



It's like this but flattened

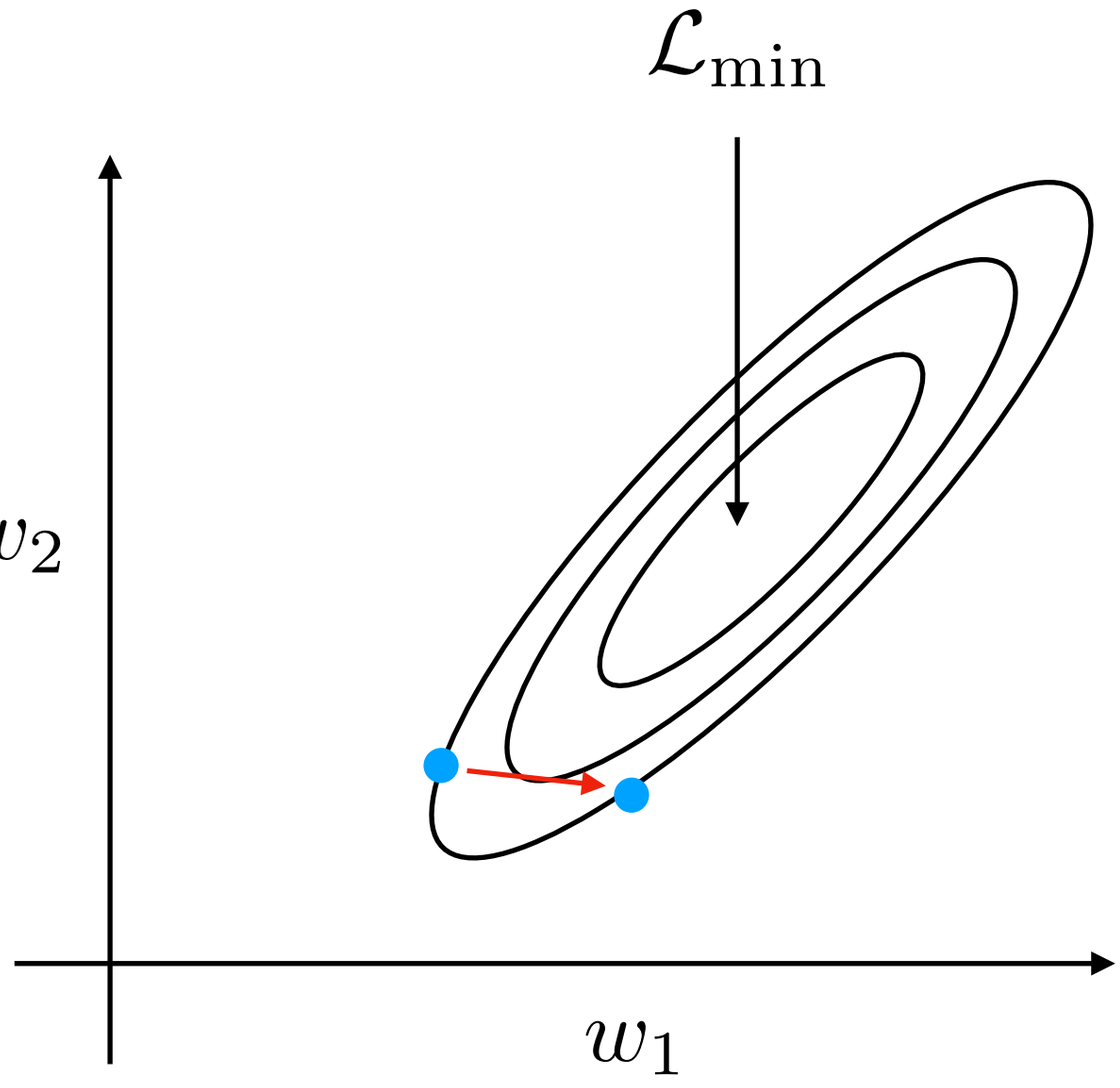


Stochastic Gradient Descent as Surface Plot



Stochastic updates
are a bit **noisier**, because
each minibatch is an approximation
of the overall loss on the
training set
(later, in deep neural nets, we
will see why noisier updates are
actually helpful)

Batch Gradient Descent as Surface Plot

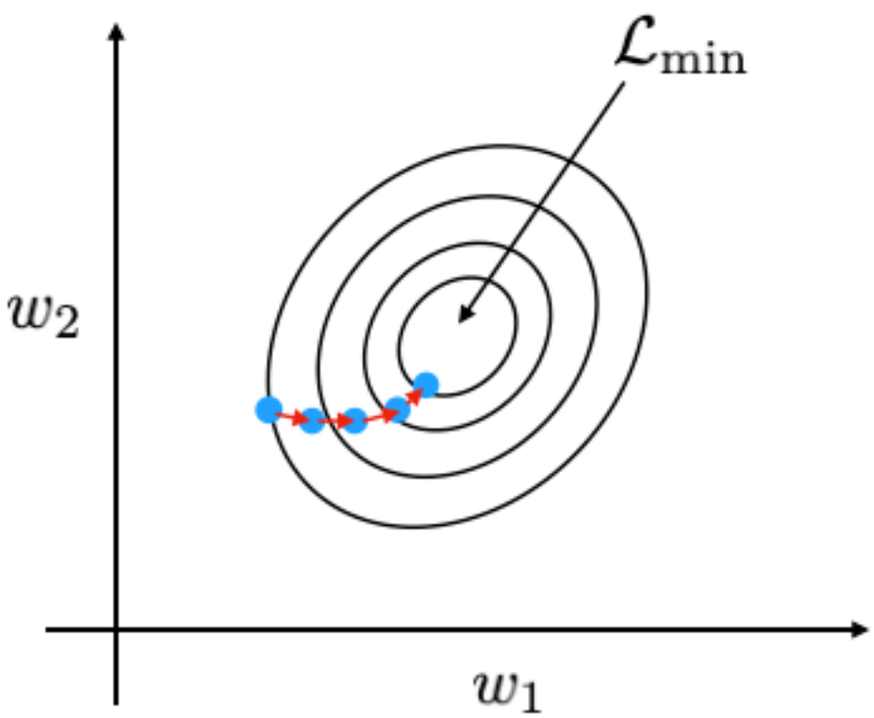


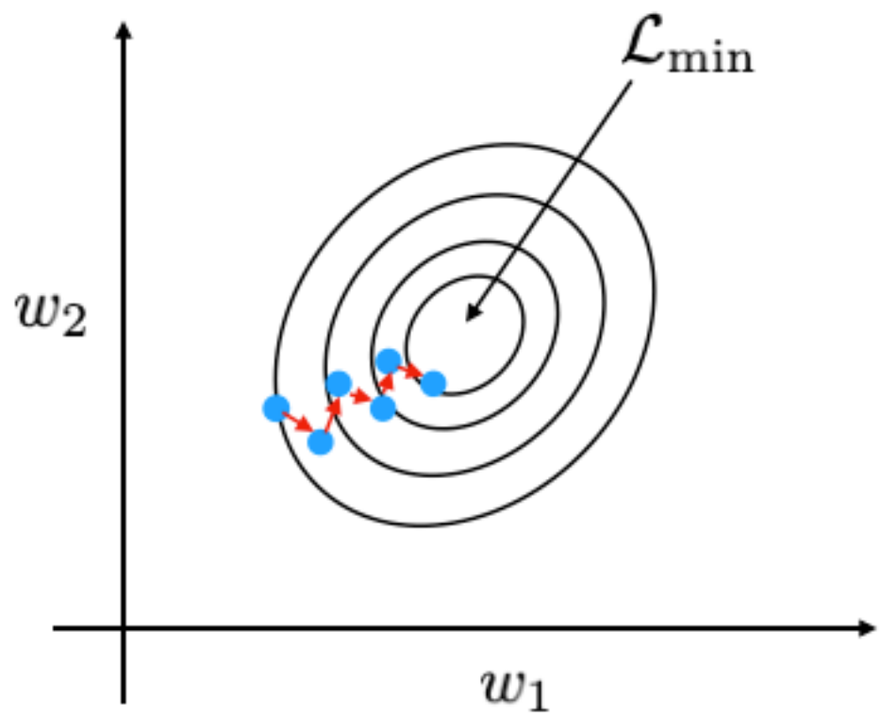
If inputs are on very different scales some weights will update more than others ... and it will also harm convergence

(always normalize inputs!)

Q: In linear regression, why do we normalize inputs?

Comparing stochastic GD with batch GD





Why is stochastic (on-line or minibatch) noisier than batch ("whole-training-set") gradient descent?

More on SGD

Why is stochastic (on-line or minibatch) noisier than batch ("whole-training-set") gradient descent?

1. Imagine you are a scientist who develops a new pharmaceutical drug.
2. You want to know its average efficiency to further improve the formula.
3. In order to know the average effectiveness, you would have to test this on all patients in the world.
4. This would be very expensive and take a long time before you get feedback! (Like "batch gradient descent").
5. Instead, you select a smaller group of patients (like a "minibatch").
6. Your estimate will be an estimate of the true average effectiveness. The larger the sample size, the better your estimate but the higher the cost; assume a certain sample size is enough such that the estimate is accurate enough that it will point you in the right direction when developing your drug...

Training a single-layer neural network with gradient descent

1. Online, batch, and minibatch mode
2. Relation between perceptron and linear regression
3. An iterative training algorithm for linear regression
4. (Optional) Calculus refresher I: Derivatives
5. (Optional) Calculus refresher II: Gradients
6. Understanding gradient descent
7. **Training an adaptive linear neuron (Adaline)**

(Least-Squares) Linear Regression

We want to speed up computation and memory when input data has large dimensions

$$\mathbf{w} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top y$$

(assuming that the bias is included in $\mathbf{w}$, and the design matrix has an additional vector of 1's)

Q: Any idea? Does random sampling work?

ADALINE

Widrow and Hoff's ADALINE (1960)

A nicely differentiable neuron model

Widrow, B., & Hoff, M. E. (1960). *Adaptive switching circuits* (No. TR-1553-1). Stanford Univ Ca Stanford Electronics Labs.

Widrow, B. (1960). *Adaptive" adaline" Neuron Using Chemical" memistors."*.

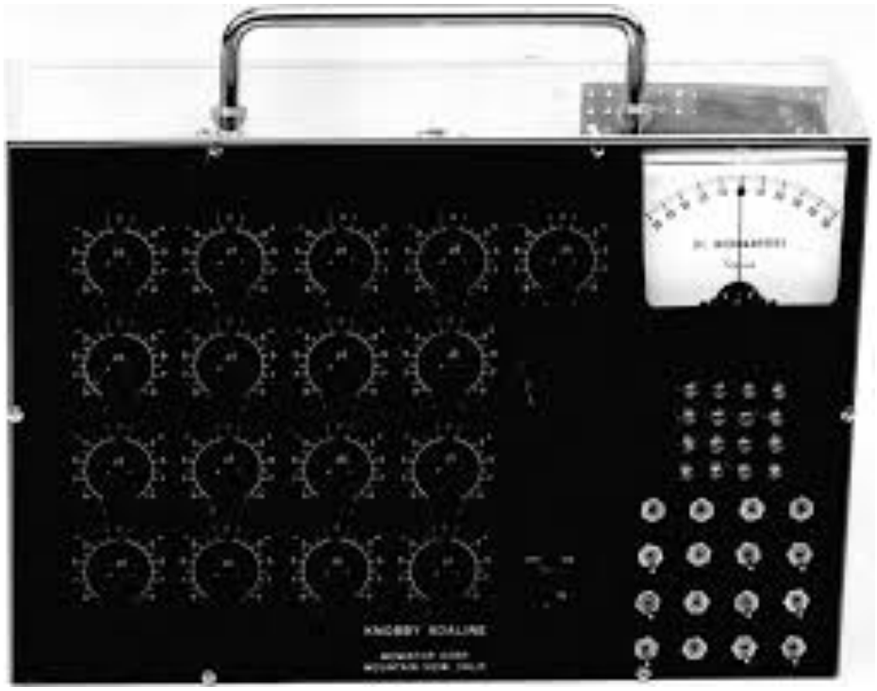
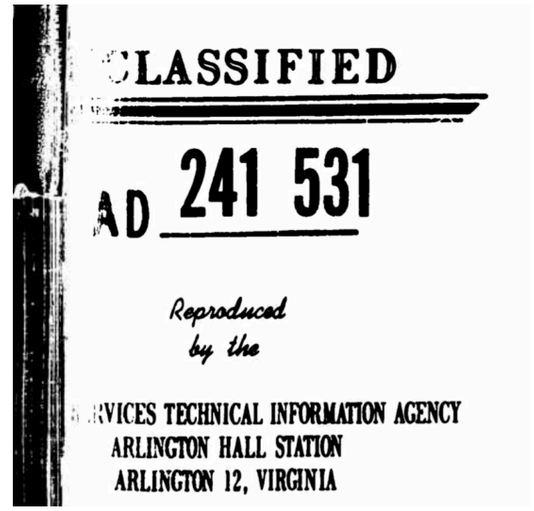
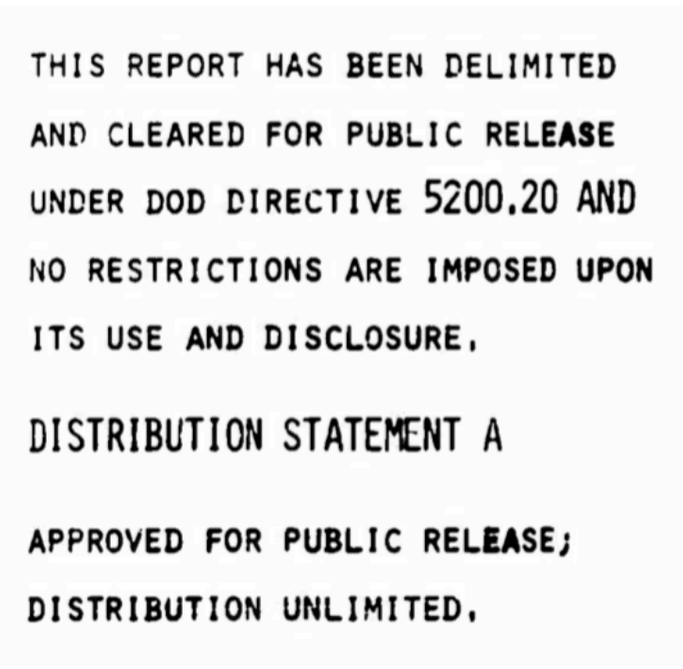


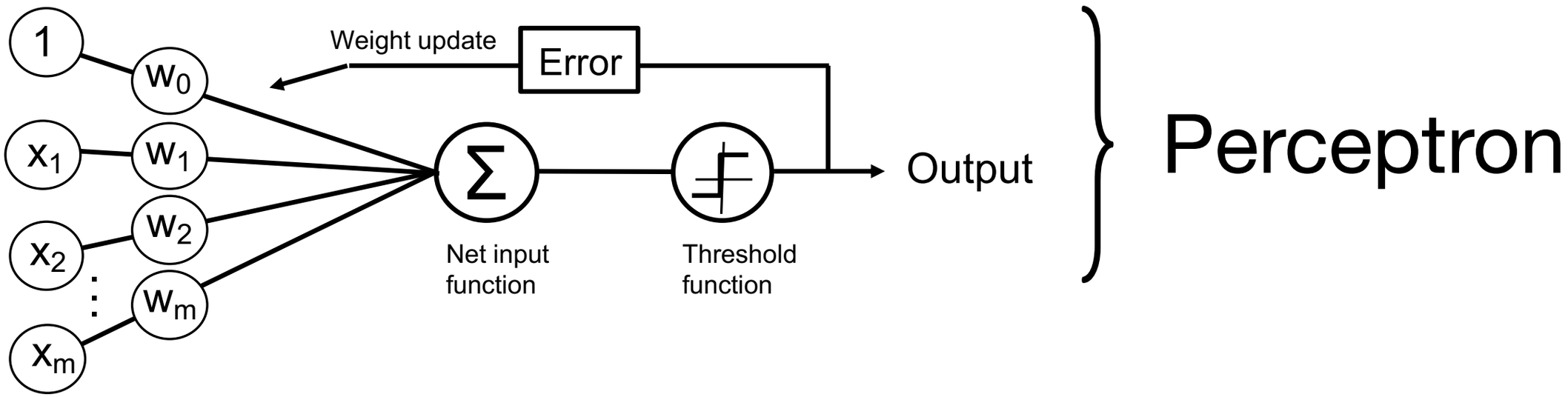
Image source: https://www.researchgate.net/profile/Alexander_Magoun2/publication/265789430/figure/fig2/AS:392335251787780@1470551421849/ADALINE-An-adaptive-linear-neuron-Manually-adapted-synapses-Designed-and-built-by-Ted.png

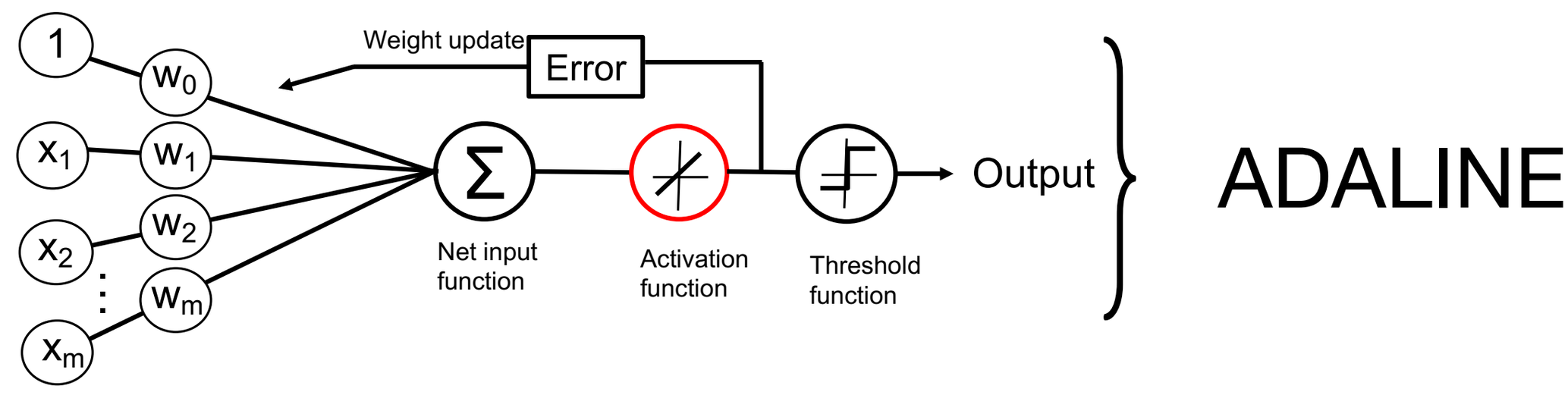




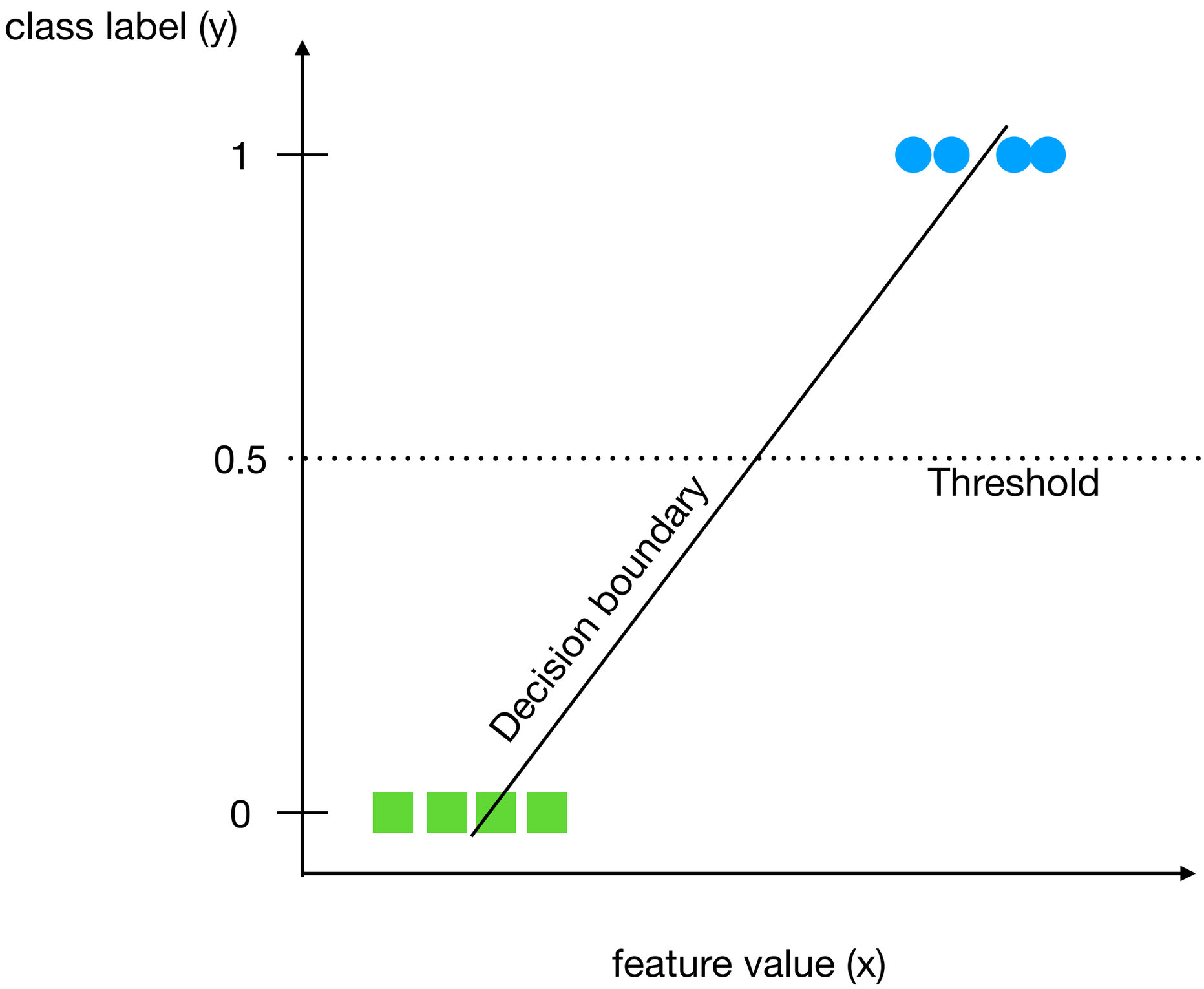
ADALINE

ADAptive LInear NEuron





Linear Regression


class label (y)
1
0.5
0
Threshold
Decision boundary
feature value (x)

Code Examples

https://github.com/rasbt/stat453-deep-learning-ss21/tree/master/L05/code

Next Lecture:

Neurons with non-linear activation functions